



Department of Aerospace Engineering University of Cincinnati

(NASA-CR-155328) NUMERICAL METHODS FOR
INVISCID/VISCOUS FLOWS (Cincinnati Univ.)
103 p HC A06/MF A01 CSCI 20D

N78-12366

Unclas

G3/34 53610

NUMERICAL METHODS FOR INVISCID/VISCOUS FLUID FLOWS

T.M. EL-MISTIKAWY

AND

M.J. WERLE

This research was supported by NASA Langley Research Center under Grant NSG 1208 and by the Air Force Office of Scientific Research through the Flight Dynamics Laboratory under Contract F33615-76-C-3091.

Approved for public release, distribution unlimited.

September 1977



NUMERICAL METHODS FOR INVISCID/VISCOUS FLUID FLOWS

T.M. El-Mistikawy^{*}

and

M.J. Werle^{**}

Aerospace Engineering and Applied Mechanics Department

University of Cincinnati

Cincinnati, Ohio

This research was supported by NASA Langley Research Center under Grant NSG 1208 and by the Air Force Office of Scientific Research through the Flight Dynamics Laboratory under Contract F33615-76-C-3091.

Approved for public release, distribution unlimited.

* Graduate Research Assistant

** Professor

NUMERICAL METHODS FOR INVISCID/VISCOUS FLUID FLOWS

T.M. El-Mistikawy and M.J. Werle

ABSTRACT

A study of numerical schemes for solving viscous fluid flow problems with sizable regions of predominantly inviscid flow is presented. Difficulties associated with the familiar central difference approach for such problems are analyzed and alternative finite difference approaches employing windward concepts are presented. In addition, difference relations based on exponential operators are developed. All such schemes are demonstrated and evaluated through application to the case of Falkner Skan flow with blowing - a problem in which a sizable region of predominantly inviscid flow develops near the injection surface that traditionally causes numerical difficulty. From these studies it is concluded that the exponential-box-scheme (EBS) developed here provides a definite advantage over the other schemes studied.

TABLE OF CONTENTS

	<u>Page</u>
I. INTRODUCTION	1
II. GOVERNING EQUATIONS	4
1. General Concepts	4
2. The Falkner Skan Equations	4
3. A Model Convection/Diffusion Problem	5
4. Numerical Concepts	6
III. FINITE DIFFERENCE OPERATORS FOR CONVECTION/DIFFUSION FLOWS	6
1. The Central Difference Scheme	6
2. Artificial Viscosity-Corrected Central Difference Scheme	11
3. Upwind Schemes	13
4. Summary and Comparison of Finite Difference Schemes	21
IV. THE METHOD OF EXPANDED COEFFICIENTS FOR CONVECTION/ DIFFUSION FLOWS	22
1. The General Concept	22
2. The Three-Point Exponential Scheme	23
3. The Exponential Box Scheme	26
V. APPLICATION TO FALKNER SKAN FLOWS WITH INJECTION	29
1. General Statement	29
2. Difference Equations	31
(a) The Central Difference Scheme	31
(b) The Corrected Central Difference Scheme	31
(c) The Midpoint Upwind Scheme	31

	<u>Page</u>
(d) Roscoe's UDR Scheme	34
(e) Three-Point Exponential Scheme	34
(f) Exponential Box Scheme	35
VI. NUMERICAL RESULTS AND DISCUSSION	38
1. General Statement	38
2. Case I - The Blasius Equation	39
3. Case II - Moderate Injection at an Axisymmetric Stagnation Point	41
4. Case III - Strong Blowing at an Axisymmetric Stagnation Point	43
VII. CONCLUSIONS AND RECOMMENDATIONS	46
VIII. REFERENCES	48
APPENDIX A - WIGGLES AND DIAGONAL DOMINANCE	50
APPENDIX B - A GENERAL THREE POINT FINITE DIFFERENCE EQUATION	53
APPENDIX C - ERROR ANALYSIS FOR EXPONENTIAL SCHEMES	55
APPENDIX D - A SECOND GENERATION EXPONENTIAL BOX SCHEME	60

ORIGINAL PAGE IS
OF POOR QUALITY

LIST OF SYMBOLS

A, B	arbitrary constants in equation (45)
a, b, c	coefficients in equation (4a)
C	constant coefficient in equation (4b)
D	arbitrary constant in equation (82)
d, e, g, h	coefficients in equation (D9) defined by equations (D10)
F	longitudinal velocity in Falkner Skan equations (1) and dependent variable in equations (4)
$\bar{F}, \bar{V}$	previous iteration or station values of F and V
f	symbol defined by equation (D2)
j	integer associated with the j th grid point used in equations (9) and (A5)
$N+1$	total number of grid points used
P	particular solution of equation (8) used in equation (9)
R_1, R_2	arbitrary constants in equation (9) and (A5)
$\bar{R}_1, \bar{R}_2$	arbitrary constants in equation (14)
r, t	symbols defined by equations (D8)
V	normal velocity in Falkner Skan equations (1)
V_w	injection velocity
α	parameter locating the general point "s" in the interval $[j-1, j+1]$
α_i	coefficients used in equation (8), $i=1,2,3,4$
α_{ij}	coefficients used in equation (61), $i=1,2$
$\alpha_{ij}, \beta_{ij}, \gamma_{ij}$	coefficients used in equations (D11, D12), $i=1,2$.

β	pressure gradient coefficient in Falkner Skan equations (1)
$\Delta\eta$	step size in the η -direction
ϵ	second order error term given by equation (B7)
η	distance normal to surface in Falkner-Skan equations (1) and independent variable in equations (4)
θ, ρ	phase angle and magnitude of complex λ , (Appendix A)
λ	characteristic root given by equation (10)
λ_1, λ_2	two characteristic roots in equation (9)
$\bar{\lambda}_1, \bar{\lambda}_2$	exact values of λ_1 and λ_2 given by equations (15)
μ_{ij}, ν_{ij}	coefficients in equations (D13, D14), $i=1,2$
ν	artificial viscosity in equation (20)

Subscripts

j	integer refers to the grid point location ($j=1, \dots, N+1$)
m, n	refer to the midpoints of the intervals $[j-1, j]$ and $[j, j+1]$ respectively
s	refers to a general point in the interval $[j-1, j+1]$
w	refers to the wall values
η	represents differentiation w.r.t. η

LIST OF TABLES

	<u>Page</u>
<u>BLASIUS FLOW</u>	
1 VELOCITY PROFILES FOR FINITE DIFFERENCE SCHEMES	65
2 VELOCITY PROFILES FOR THE EXPONENTIAL SCHEMES	65
<u>MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT</u>	
3 VELOCITY PROFILES FOR FINITE DIFFERENCE SCHEMES	66
4 VELOCITY PROFILES FOR THE EXPONENTIAL SCHEMES	66
<u>STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT</u>	
5 VELOCITY PROFILES FOR FINITE DIFFERENCE SCHEMES	67
6 VELOCITY PROFILES FOR THE EXPONENTIAL SCHEMES	68

ORIGINAL PAGE IS
OF POOR QUALITY

LIST OF FIGURES

	<u>Page</u>
1 VISCID/INVISCID FLUID FLOW - FLOW OVER A REENTRY BODY . . .	69
2 THREE-POINT UNIFORM GRID CONFIGURATION	70
3 CHARACTERISTIC ROOT λ_2 OF FINITE DIFFERENCE SCHEMES . . .	71

BLASIUS FLOW

4 VELOCITY PROFILE	72
5 MOMENTUM BALANCE	73
6 S.S.S.* FOR FINITE DIFFERENCE SCHEMES - F AT $\eta=1.0$	74
7 S.S.S. FOR THE EXPONENTIAL SCHEMES - F AT $\eta=1.0$	75
8 S.S.S. FOR FINITE DIFFERENCE SCHEMES - F AT $\eta=2.0$	76
9 S.S.S. FOR THE EXPONENTIAL SCHEMES - F AT $\eta=2.0$	77

MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT

10 VELOCITY PROFILE	78
11 MOMENTUM BALANCE	79
12 S.S.S. FOR FINITE DIFFERENCE SCHEMES - F AT $\eta=6.0$	80
13 S.S.S. FOR THE EXPONENTIAL SCHEMES - F AT $\eta=6.0$	81
14 S.S.S. FOR FINITE DIFFERENCE SCHEMES - F AT $\eta=8.0$	82
15 S.S.S. FOR THE EXPONENTIAL SCHEMES - F AT $\eta=8.0$	83
16 S.S.S. FOR FINITE DIFFERENCE SCHEMES - F AT $\eta=10.0$	84
17 S.S.S. FOR THE EXPONENTIAL SCHEMES - F AT $\eta=10.0$	85

* S.S.S. = Step Size Study.

	<u>Page</u>
<u>STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT</u>	
18 VELOCITY PROFILE	86
19 MOMENTUM BALANCE	87
20 S.S.S. FOR FINITE DIFFERENCE SCHEMES - F AT $\eta=48$. . .	88
21 S.S.S. FOR THE EXPONENTIAL SCHEMES - F AT $\eta=48$. . .	89
22 S.S.S. FOR FINITE DIFFERENCE SCHEMES - F AT $\eta=50$. . .	90
23 S.S.S. FOR THE EXPONENTIAL SCHEMES - F AT $\eta=50$. . .	91
24 S.S.S. FOR FINITE DIFFERENCE SCHEMES - F AT $\eta=52$. . .	92
25 S.S.S. FOR THE EXPONENTIAL SCHEMES - F AT $\eta=52$. . .	93

ORIGINAL PAGE IS
OF POOR QUALITY

I. INTRODUCTION

The subject of this study are those numerical methods used for solving viscous fluid flow problems with particular interest centered on problems where there exists distinct inviscid and viscous regions within the calculation domain. For example, as depicted in Figure 1, within the supersonic flow field over a reentry body where surface ablation occurs, an inviscid region is formed near the body surface under a "blown off" boundary layer, which itself is driven by another inviscid region close to the bow shock wave. To date, finite difference schemes for numerical solution of the governing partial differential equations have encountered difficulty with this change in flow character apparently due to the diffusion character of their difference operators. This difficulty becomes especially troublesome when implicit finite difference schemes are employed. Although such schemes suffer no integration step size limitation for stability sake, they are found (see Roache, Ref. 1) to be restricted by "accuracy" requirements to virtually the same step size ranges as explicit finite difference schemes. Thus a need exists to remove this limitation.

Numerical schemes based on central difference concepts are found to produce large truncation errors of an oscillatory type that do not belong to the physical problem, but rather to the difference equations themselves (see Hirsh and Rudy, Ref. 2 and Roache, Ref. 1). For want of a better term, Roache (Ref. 1) called these oscillations "wiggles" and he gave an explanation

for their occurrence based on asymptotic concepts and he described a mechanism in which these oscillations occur in a simple convection-diffusion problem. The basic difficulty appears to be centered in the difference representation of the convective process and no completely satisfactory remedy⁸ is proposed. One solution to this difficulty is based on the use of an active stepsize control procedure as discussed by Roache and used by Liu and Chiu (Ref. 3). Alternatively, artificial viscosity like terms can be injected in the difference equation to eliminate these oscillations [e.g. Briley and MacDonald, Ref. 4] or one can employ upwind differencing for representing the convection terms in the governing equations [for examples see the methods used by Garrett, Smith and Perkins (Ref. 5) or by Raithby (Ref. 6)]. In addition, it has recently been found that numerical schemes based on series expansions of the coefficients appearing in the governing differential equations result in exponential operators that are inherently devoid of this oscillatory difficulty. Such schemes were recently proposed by Spalding^{*} (Ref. 7) and were used by Roscoe (Refs. 10 and 11) and Chien (Ref. 12) for solving the two and three dimensional Navier-Stokes equations for a simple geometry.

* A more general approach leading to the use of exponential operators was presented by Pruess (Ref. 8) and later by Rose (Ref. 9).

In the present work, a detailed study of the problem of numerical oscillations and the different methods used to overcome this difficulty is carried out. Finite difference schemes based on the addition of artificial viscosity and the use of second order accurate windward differences are presented. In addition, schemes based on the method of expanded coefficients (exponential schemes) are studied and a new box version of same is presented. Application of five difference methods to the solution of the Falkner Skan equations with surface injection show a clear indication of the superiority of one such method (here termed the exponential box scheme) for such problems.

II. GOVERNING EQUATIONS

1. General Concepts

In the numerical solution of the viscous/inviscid fluid flow problems by implicit techniques, the governing equations, which in general are partial differential equations, are first reduced to two point boundary value problems in a direction perpendicular to the principal flow direction. This is, for example, how the non self similar two dimensional boundary layer equations are solved. Also, an ADI solution of Navier-Stokes equations adopts the same treatment. Attention then need initially only be focused on the development of numerical techniques for ordinary differential equations with split boundary conditions.

2. The Falkner-Skan Equations

A model for the ordinary differential equations resulting from this treatment of the viscid/inviscid fluid flow problems are the mass injection version of Falkner-Skan equations for the self similar boundary layer given here as

$$V_{\eta} + F = 0 \quad (1a)$$

$$F_{\eta\eta} - VF_{\eta} - \beta F^2 = -\beta \quad (1b)$$

where F is the longitudinal velocity, V is the normal velocity and β is the pressure gradient coefficient. The boundary conditions for present interest are given as

$$F(0) = 0 \quad \text{and} \quad V(0) = V_w \quad (2a)$$

$$\text{with} \quad F(\eta) \rightarrow 1 \quad \text{as} \quad \eta \rightarrow \infty \quad (2b)$$

The first term in the momentum equation (1b) represents the diffusion effect while the second term represents normal convection effects. The other two terms represent the longitudinal convection and pressure gradient effects. However in this one dimensional formulation of the problem they appear as source terms.

In regions where the flow is basically inviscid (such as near a wall with large blowing) the diffusion effects are expected to disappear and a model of the resulting ordinary differential equation representing the momentum balance will be

$$-VF_{\eta} - \beta F^2 = -\beta \quad (3)$$

A numerical scheme that is used for solving fluid flow problems where there are transitions from inviscid to viscous regions should be able to accommodate this change in the character of the governing equations.

3. A Model Diffusion-Convection Problem

A general differential equation that involves both the diffusion and convection effects can be put in the form

$$F_{\eta\eta} - (a+b)F_{\eta} + abF = c \quad (4a)$$

In light of the preceding section, equation (4a) can be looked at as a momentum conservation equation where the first term represents diffusion effects while the second, normal convection effects. For initial interest, only normal convection effects will be considered by taking $b = c = 0$ and $a = C$, a constant. Thus

$$F_{\eta\eta} - C F_{\eta} = 0 \quad (4b)$$

4. Numerical Concepts

There are two methods that can be used to represent the differential equations by difference equations. In the first, the finite difference method (Ref.13), difference operators are obtained by a Taylor's series expansion approach and these are used to replace the differential operators that appear in the governing equations. This method is straightforward and provides a formal order of accuracy for each such scheme in the limit of small step size. In the second, the expanded coefficient method, the differential equations are formally integrated over a small interval in which its coefficients are approximated by Taylor's series representations resulting in a new system of difference equations generally involving exponential operators.

III. FINITE DIFFERENCE OPERATORS FOR CONVECTION/DIFFUSION FLOWS

1. The Central Difference Scheme

Consider a portion of the flow field depicted in Figure 2 with $j-1$, j , and $j+1$ points of a grid of uniform spacing $\Delta\eta$, and let "m" be the midpoint between the $j-1$ and j th point and "n" the midpoint between the j and $j+1$ th point.

To obtain the central difference representation of equation (4b) one can use the Taylor's series expansion to evaluate the derivatives F_{η} and $F_{\eta\eta}$ at the j th point in the central difference form

$$F_{\eta j} = \frac{F_{j+1} - F_{j-1}}{2\Delta\eta} - \frac{1}{6} \Delta\eta^2 F_{\eta\eta\eta j} + O(\Delta\eta^4) \quad (5)$$

$$\text{and } F_{nnj} = \frac{F_{j+1} - 2F_j + F_{j-1}}{\Delta n^2} - \frac{1}{12} \Delta n^2 F_{nnnnj} + O(\Delta n^4) \quad (6)$$

so that upon evaluating equation (4b) at the j th point and neglecting terms $O(\Delta n^2)$, one obtains the central difference equation

$$(1 - \frac{1}{2} \Delta n C) F_{j+1} - 2F_j + (1 + \frac{1}{2} \Delta n C) F_{j-1} = 0 \quad (7)$$

The exact solution of a difference equation of the form

$$\alpha_1 F_{j+1} + \alpha_2 F_j + \alpha_3 F_{j-1} = \alpha_4 \quad (8)$$

can be written as

$$F_j = R_1 \lambda_1^j + R_2 \lambda_2^j + P_j \quad (9)$$

where λ_1 and λ_2 are solutions of the characteristic equation

$$\alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3 = 0 \quad (10)$$

R_1 and R_2 are arbitrary constants determined by the boundary conditions and P_j is a particular solution of equation (8).

For equation (7)

$$\alpha_1 = 1 - \frac{1}{2} \Delta n C \quad (11a)$$

$$\alpha_2 = -2 \quad (11b)$$

$$\alpha_3 = 1 + \frac{1}{2} \Delta n C \quad (11c)$$

$$\alpha_4 = 0 \quad (11d)$$

so that

$$\lambda_1 = 1 \quad (12a)$$

$$\lambda_2 = \frac{1 + \frac{1}{2} \Delta n C}{1 - \frac{1}{2} \Delta n C} \quad (12b)$$

$$\text{and } P_j = 0 \quad (12c)$$

Thus the exact solution of the central difference equation (7) can be written as

$$F_j = R_1 + R_2 \lambda_2^j \quad (13)$$

where λ_2 is given by equation (12b).

The exact values of λ_1 and λ_2 can be obtained by comparing equation (9) with the exact solution of the differential equation (4b) evaluated at the grid points, namely,

$$F_j = \bar{R}_1 + \bar{R}_2 e^{j\Delta n C} \quad (14)$$

so that the exact values of the characteristic roots are given as

$$\bar{\lambda}_1 = 1 \quad (15a)$$

$$\bar{\lambda}_2 = e^{\Delta n C} \quad (15b)$$

Note that in equation (12b) λ_2 will be negative when

$$\Delta n |C| > 2 \quad (16)$$

whereas its exact counterpart, $\bar{\lambda}_2$ is always positive. In such a case, λ_2^j will change its sign according to whether j is even or odd and hence the solution (13) will show an oscillatory behavior compared to the monotonic behavior of the exact solution (14).

This oscillatory behavior has been observed in the central difference solution of the Falkner-Skan equations near the outer edge of the boundary layer and in many other problems when a condition similar to condition (16) and obtained in the same way is locally satisfied.

Thus it is seen that these oscillations belong to the difference equations and are a part of their exact solution. This eliminates the round off errors due to the lack of diagonal dominance from being a cause of these oscillations. Roache (Ref. 1) attributes these oscillations (wiggles) to a singular behavior of the difference equation (7) when the cell Reynolds number $\Delta\eta|C|$ becomes large ($\Delta\eta|C| > 2$) compared to the singular behavior of the differential equation (4b) in the limit when the effective Reynolds number $|C|$ becomes infinitely large.

To give a physical interpretation to this "unstable" behavior of the difference operator, it is convenient to employ a control volume (or conservation) representation of these equations. The central difference equations (7) can be obtained through a control volume approach by first integrating equation (4b) over the control volume $[m,n]$ to get

$$(F_{\eta_n} - F_{\eta_m}) - C(F_n - F_m) = 0 \quad (17)$$

and then using the central difference representations

$$F_{\eta_m} = \frac{F_j - F_{j-1}}{\Delta\eta} + O(\Delta\eta^2) \quad (18a)$$

$$F_{\eta_n} = \frac{F_{j+1} - F_j}{\Delta\eta} + O(\Delta\eta^2) \quad (18b)$$

$$F_m = \frac{F_j + F_{j-1}}{2} + O(\Delta\eta^2) \quad (18c)$$

$$F_n = \frac{F_{j+1} + F_j}{2} + O(\Delta\eta^2) \quad (18d)$$

Note that in this representation of the convection terms there is an explicit downstream contribution to the momentum, carried by the flow as it moves with the velocity C . This downstream contribution is a characteristic of a central difference representation of the convection effects that is expected to cause large errors and instabilities of the scheme.

In fact, the oscillatory behavior described in the previous section is how these instabilities manifest themselves. Note that according to condition (16), the oscillatory behavior is encountered when the step size $\Delta\eta$ is large in which case the downstream contribution to the convection effects comes from far points, or when the normal velocity C is large in which case the downstream contribution itself becomes large.

Condition (16) for the oscillatory behavior suggests the control of the step size so that

$$\Delta\eta |C| \leq 2 \quad (19)$$

for no oscillations.

For the Falkner Skan equations where the normal velocity V varies from point to point a condition equivalent to condition (19) should be locally satisfied. Thus, in this case the control of the step size can take one of the following forms:

1. using, with a uniform grid, a step size that uniformly satisfies the condition for no oscillations. Thus even in regions where larger step sizes can be used, the step size should be less than or equal to that corresponding to the severest case. This means a larger number of mesh points and consequently greater computational time and storage requirements.

2. using a variable step size that locally satisfies the condition for no oscillations as used by Liu and Chiu (Ref. 3) for massive blowing problems. Actually use of Keller's box scheme (see Blottner, Ref. 14), which is a version of the central difference approach is directly adaptive to this method. However such an approach is unsuitable to nonlinear equations since in an iterative procedure the solution changes from one iteration to another.

Two other approaches can be used. The first is the use of non-centered differencing for the convection terms which will be discussed in a later section. The second is the addition of explicit artificial viscosity terms to the governing equations.

2. Artificial Viscosity - Corrected Central Difference Scheme

In the second method terms of the form $\nu F_{\eta\eta}$ are added to the differential equation, e.g. equation (4b) becomes

$$(1+\nu) F_{\eta\eta} - CF_{\eta} = 0 \quad (20)$$

with the central difference representation given by

$$(1+\nu) \frac{F_{j+1} - 2F_j + F_{j-1}}{\Delta\eta^2} - C \frac{F_{j+1} - F_{j-1}}{2\Delta\eta} = 0 \quad (21)$$

whose characteristic roots are

$$\lambda_1 = 1 \quad (22a)$$

$$\lambda_2 = \frac{1 + v + \frac{1}{2} \Delta \eta C}{1 + v - \frac{1}{2} \Delta \eta C} \quad (22b)$$

So, the condition for no oscillations will be

$$\Delta \eta |C| \leq 2(1+v) \quad (23)$$

The artificial viscosity v is usually constructed to be of higher order so that the accuracy of the method remains second order.

A more consistent method to introduce this artificial viscosity, can be based on a second order correction to the central difference representation of the derivatives. In equations (5) and (6) one can relate $F_{\eta\eta\eta_j}$ and $F_{\eta\eta\eta\eta_j}$ to $F_{\eta\eta_j}$ through differentiation of equation (4b) to get upon substitution in equation (4b) evaluated at the j th point

$$\left(1 + \frac{1}{12} \Delta \eta^2 C^2\right) \frac{F_{j+1} - 2F_j + F_{j-1}}{\Delta \eta^2} - C \frac{F_{j+1} - F_{j-1}}{2\Delta \eta} = 0 \quad (24)$$

Hence, comparing with equation (21) the correction introduced is equivalent to an artificial viscosity

$$v = \frac{1}{12} \Delta \eta^2 C^2$$

which is second order in $\Delta \eta$ and hence does not harm the order of accuracy of the scheme.

The difference equation (24) can be put in the form

$$\begin{aligned} \left[1 - \frac{1}{2} \Delta \eta C + \frac{1}{12} \Delta \eta^2 C^2\right] F_{j+1} - \left[2 + \frac{1}{6} \Delta \eta^2 C^2\right] F_j \\ + \left[1 + \frac{1}{2} \Delta \eta C + \frac{1}{12} \Delta \eta^2 C^2\right] F_{j-1} = 0 \end{aligned} \quad (25)$$

which has the characteristic roots

$$\lambda_1 = 1 \quad (26a)$$

$$\lambda_2 = \frac{(1 + \frac{1}{4} \Delta\eta C)^2 + \frac{1}{48} \Delta\eta^2 C^2}{(1 - \frac{1}{4} \Delta\eta C)^2 + \frac{1}{48} \Delta\eta^2 C^2} \quad (26b)$$

Thus λ_2 is always positive and the solution is non oscillatory. From equations (24) and (26b), it is seen that the smallest amount of artificial viscosity proportional to $\Delta\eta^2 C^2$ that can be added to prevent the oscillations is $\frac{1}{16} \Delta\eta^2 C^2$ which can be looked at as that part of the truncation error that is responsible for creating the oscillations. This suggests that the L.H.S. of equation (24) can be used to represent the differential expression $F_{\eta\eta_j} - CF_{\eta_j}$ whenever it appears in a differential equation instead of introducing a complete correction to the difference equation which requires differentiation.

3. Upwind Schemes

The analysis of the previous sections has shown how a central difference representation of the convection terms can produce the problem of the oscillatory behavior of the numerical solution, and has related that to the explicit downstream contribution to the convection effects.

The idea behind upwinding is to correctly represent the convection terms so that the information carried by the flow as it moves is the upstream information only.

A first order accurate upwind representation of equation (4b) can be obtained if in equation (17) the diffusion terms are represented by equations (18a, b) and the convection terms are represented by

$$F_m = F_{j-1} + O(\Delta\eta) \quad (27a)$$

and $F_n = F_j + O(\Delta\eta) \quad (27b)$

when $C > 0$, i.e. the flow is upward, and by

$$F_m = F_j + O(\Delta\eta) \quad (28a)$$

and $F_n = F_{j+1} + O(\Delta\eta) \quad (28b)$

when $C < 0$, i.e. the flow is downward. The resulting difference equation can be put in the form

$$\left[1 - \frac{1}{2} \Delta\eta(C - |C|)\right] F_{j+1} - [2 + \Delta\eta|C|] F_j + \left[1 + \frac{1}{2} \Delta\eta(C + |C|)\right] F_{j-1} = 0 \quad (29)$$

which can be also obtained through the Taylor's series expansion approach if equation (4b) is evaluated at the j th point and a central difference representation of $F_{\eta\eta_j}$ and the following upwind difference representations of F_{η_j} are used

$$F_{\eta_j} = \frac{F_j - F_{j-1}}{\Delta\eta} + O(\Delta\eta) \quad \text{if } C > 0 \quad (30a)$$

$$F_{\eta_j} = \frac{F_{j+1} - F_j}{\Delta\eta} + O(\Delta\eta) \quad \text{if } C < 0 \quad (30b)$$

The characteristic roots of equation (29) are

$$\lambda_1 = .1 \quad (31a)$$

and
$$\lambda_2 = \frac{1 + \frac{1}{2} \Delta \eta (C + |C|)}{1 - \frac{1}{2} \Delta \eta (C - |C|)} \quad (31b)$$

In both cases $C > 0$ and $C < 0$, λ_2 is positive and the solution is nonoscillatory which supports the upwinding idea.

Applied to the Falkner Skan momentum equation, where the normal velocity V changes from point to point and may even change its direction, the first order accurate upwind scheme described above will have to follow the direction of V and to switch when its direction is changed so that the information carried by the flow is always the upstream information.

Another approach to obtain a first order accurate upwind representation of equation (4b) is to evaluate this equation at point "m" when $C > 0$ and to use the following second order accurate representation of F_{η_m} and first order accurate representation of $F_{\eta\eta_m}$

$$F_{\eta_m} = \frac{F_j - F_{j-1}}{\Delta \eta} + O(\Delta \eta^2) \quad (32a)$$

$$F_{\eta\eta_m} = \frac{F_{j+1} - 2F_j + F_{j-1}}{\Delta \eta^2} + O(\Delta \eta) \quad (32b)$$

When $C < 0$ equation (4b) is evaluated at point "n" and the following representation of F_{η_n} and $F_{\eta\eta_n}$ are used

ORIGINAL PAGE IS
OF POOR QUALITY

$$F_{\eta n} = \frac{F_{j+1} - F_j}{\Delta \eta} + O(\Delta \eta^2) \quad (33a)$$

$$F_{\eta \eta n} = \frac{F_{j+1} - 2F_j + F_{j-1}}{\Delta \eta^2} + O(\Delta \eta) \quad (33b)$$

In the case of equation (4b) this approach will lead to the same difference equation (29) as the previous approach. However this will not be true if C is variable or in the case of a nonlinear equation like the Falkner Skan momentum equation. In fact, there are two main differences between the previous two approaches. The first difference is in the order of accuracy in which the diffusion and convection terms are represented. For the first approach, the diffusion terms are second order accurate while the convection terms are first order accurate. For the second approach the diffusion terms are first order accurate while the convection terms are second order accurate. Thus in a region of inviscid flow where the diffusion term disappears from the momentum equation the second approach will lead to a second order accurate scheme while the first approach will lead to a first order accurate scheme. The second difference is in the control volume for which the difference equation is the momentum balance equation. For the first approach the control volume is the interval between the two midpoints "m" and "n". For the second approach it is the interval between $j-1$ and j if $C > 0$ and the interval between j and $j+1$ if $C < 0$.

A special treatment at the switch region should take place to prevent excessive errors due to the nonconservative nature

of these two upwind schemes. This is especially required for the second approach where the momentum conservation for a control volume near the switch point is not considered.

Corrections to the first order upwind schemes described above to be second order accurate can be introduced by considering the next term in the Taylor's series expansion representation for the first order accurate derivative involved. These corrections can be introduced either in an explicit sense using values from the previous iteration or in an implicit sense.

In the discussion that follows, only the case when $C > 0$ will be considered. Similar treatment can be applied to the case when $C < 0$. Consider first the windward approach that corrects the first order representation of the windward convection operators given by equations (30a) and (30b). For this approach note that the first derivative $F_{\eta j}$ can be expressed as

$$F_{\eta j} = F_{\eta m} + \frac{1}{2} \Delta \eta F_{\eta \eta m} + O(\Delta \eta^2) \quad (34)$$

In equation (30a), only the first term was considered and was represented with second order accuracy. To get a second order representation of $F_{\eta j}$, the second term should also be considered. However, since in that term $F_{\eta \eta m}$ is multiplied by $\Delta \eta$ one can, with first order error, evaluate $F_{\eta \eta}$ at any close point and still have second order accurate representation of $F_{\eta \eta j}$. One choice is to evaluate $F_{\eta \eta}$ at the $(j-1)$ th point. This leads to

$$F_{\eta j} = \frac{F_j - F_{j-1}}{\Delta \eta} + \frac{1}{2} \Delta \eta \frac{\bar{F}_j - 2\bar{F}_{j-1} + \bar{F}_{j-2}}{\Delta \eta^2} + O(\Delta \eta^2) \quad (35)$$

where central differences have been used to represent $F_{\eta\eta(j-1)}$ and the "barred" values refer to the previous iteration or station values (see Atias, Wolfshtein and Israeli, Ref. 15).

This method is consistent with the upwind idea in that upstream information is used to evaluate the correction, however, a special treatment near the end points will be needed.

A second method is to evaluate $F_{\eta\eta}$ at the j th point which leads to

$$F_{\eta j} = \frac{F_j - F_{j-1}}{\Delta\eta} + \frac{1}{2} \Delta\eta \frac{\bar{F}_{j+1} - 2\bar{F}_j + \bar{F}_{j-1}}{\Delta\eta^2} + O(\Delta\eta^2) \quad (36)$$

Eq. (36) was obtained by Khosla and Rubin (Ref. 16) by rearranging the central difference representation of $F_{\eta j}$ [equation (5)]. A scheme based on this method will have the same magnitude and type of error as the central difference scheme within the converged solution since they are basically identical in representing the derivatives $F_{\eta j}$ and $F_{\eta\eta j}$. Thus this method will give an oscillatory solution if the central difference solution is oscillatory. However this scheme has the advantage over the central difference scheme that it is always diagonally dominant (see Appendix A).

A third method is to use the governing equation to represent $F_{\eta\eta_m}$ in terms of the other terms appearing in the differential equation. From equation (4b) one can write

$$F_{\eta\eta_m} = C F_{\eta_m}$$

Substituting into equation (34) one gets

$$F_{\eta j} = (1 + \frac{1}{2} \Delta\eta C) F_{\eta_m} + O(\Delta\eta^2)$$

In difference form

$$F_{\eta j} = (1 + \frac{1}{2} \Delta \eta C) \frac{F_j - F_{j-1}}{\Delta \eta} + O(\Delta \eta^2) \quad (37)$$

which leads to the difference equation

$$\frac{1}{1 + \frac{1}{2} \Delta \eta C} \frac{F_{j+1} - 2F_j + F_{j-1}}{\Delta \eta^2} - C \frac{F_j - F_{j-1}}{\Delta \eta} = 0 \quad (38)$$

This is termed the grid point upwind scheme.

Note that, with this correction, the so-called implicit artificial viscosity associated with the first order upwind scheme has been removed by reducing the effective viscosity in the difference equation. This suggests that the L.H.S. of equation (38) can be used to represent the differential expression $F_{\eta \eta j} - CF_{\eta j}$ when $C > 0$ whenever it appears in a differential equation. This can be shown to be second order accurate.

Considering the two cases $C > 0$ and $C < 0$ a combined difference equation for equation (4b) can be put in the form:

$$\begin{aligned} & [1 - \frac{1}{2} \Delta \eta (C - |C|) + \frac{1}{8} \Delta \eta^2 (C - |C|)^2] F_{j+1} - [2 + \Delta \eta |C| + \frac{1}{2} \Delta \eta^2 C^2] F_j \\ & + [1 + \frac{1}{2} \Delta \eta (C + |C|) + \frac{1}{8} \Delta \eta^2 (C + |C|)^2] F_{j-1} = 0 \end{aligned} \quad (39)$$

whose characteristic roots are

$$\lambda_1 = 1 \quad (40a)$$

$$\lambda_2 = \frac{1 + \frac{1}{2} \Delta \eta (C + |C|) + \frac{1}{8} \Delta \eta^2 (C + |C|)^2}{1 - \frac{1}{2} \Delta \eta (C - |C|) + \frac{1}{8} \Delta \eta^2 (C - |C|)^2} \quad (40b)$$

λ_2 in equation (40b) is always positive. Moreover if $C > 0$

then $\lambda_2 > 1$. While if $C < 0$ then $\lambda_2 < 1$ which is in accordance with the behavior of $\bar{\lambda}_2$ in equation (15b).

Consider now the second windward approach that evaluated the governing equation at point "m" (or "n") using a second order accurate representation of the convection term and a first order accurate representation of the diffusion term. To correct this approach to second order note that the second derivative $F_{\eta\eta_m}$ is given, to second order accuracy by

$$F_{\eta\eta_m} = F_{\eta\eta_j} - \frac{1}{2} \Delta\eta F_{\eta\eta\eta_j} + O(\Delta\eta^2) \quad (41)$$

where, again, a first order accurate representation of $F_{\eta\eta\eta_j}$ will still lead to a second order accurate representation of $F_{\eta\eta_m}$.

Taking

$$F_{\eta\eta\eta_j} = F_{\eta\eta\eta_m} + O(\Delta\eta)$$

and using the derivative of the differential equation evaluated at point "m" to evaluate $F_{\eta\eta\eta_m}$ one gets, in the case of equation (4b),

$$F_{\eta\eta_m} = \frac{1}{1 + \frac{1}{2} \Delta\eta C} F_{\eta\eta_j} + O(\Delta\eta^2)$$

With central difference representation of $F_{\eta\eta_j}$ this becomes

$$F_{\eta\eta_m} = \frac{1}{1 + \frac{1}{2} \Delta\eta C} \frac{F_{j+1} - 2F_j + F_{j-1}}{\Delta\eta^2} + O(\Delta\eta^2) \quad (42)$$

and the difference equation (termed the midpoint upwind scheme) will be

$$\frac{1}{1 + \frac{1}{2} \Delta \eta C} \frac{F_{j+1} - 2F_j + F_{j-1}}{\Delta \eta^2} - C \frac{F_j - F_{j-1}}{\Delta \eta} = 0 \quad (43)$$

which is the same as equation (38) and hence equations (39) and (40) will apply. However, this is not true for more general differential equations than (4b); i.e. the difference equations will in general, be different.

Note that a family of three point finite difference equations can be obtained by applying the same procedure described above but for a general point rather than the midpoints (see Appendix B for details).

4. Summary and Comparison of Finite Difference Schemes

An assessment of the numerical schemes described above can be made by comparing the characteristic root λ_2 for each scheme with the exact value $\bar{\lambda}_2$ as given by equation (15b).

In Figure (3), λ_2 for the central difference (C), corrected central difference (CC), and the second order upwind (UW) schemes as given by equations (12b), (26b), and (40b) respectively and $\bar{\lambda}_2$ of the exact solution are presented.

It is clear that for small $\Delta \eta |C|$ all schemes have reasonable agreement with the exact solution. This agreement is best in the case of the corrected central difference scheme which is fourth order accurate for this special problem.

For $\Delta \eta |C| > 2$ the central difference scheme has negative λ_2 which produces the oscillatory solution and it has a singularity

at $\Delta\eta C = 2$. λ_2 for the corrected central difference scheme is always positive however when $\Delta\eta C = \pm\sqrt{12}$ λ_2 reaches an extremum after which it asymptotically approaches ± 1 to $\pm\infty$. Thus this scheme is expected to produce large errors for $\Delta\eta|C| > \sqrt{12}$.

The two branches of the upwind scheme (for positive and negative C) each follows the behavior of the exact λ_2 in its region of application. On the other hand, out of their region of application [designated by the downwind scheme (DW)] they encounter large errors and would not be reliable.

IV. THE METHOD OF EXPANDED COEFFICIENTS FOR CONVECTION/DIFFUSION FLOWS

1. The General Concept

Consider the general second order differential equation given by equation (4a) as

$$F_{\eta\eta} - (a+b) F_{\eta} + abF = c \quad (44)$$

where a , b , and c are in general functions of η and F . Instead of using Taylor's series expansions to obtain finite difference representations for F , F_{η} , and $F_{\eta\eta}$, one can instead expand the coefficients a , b , and c about a point in the interval under consideration. The first terms in these expansions are the values of those coefficients at that point. The differential equation that results if only these first terms are kept will be an ordinary differential equation with constant coefficient

that can be solved analytically. One can use this exact solution to obtain finite difference equations as will be described in the next sections.

2. The Three-Point Exponential Scheme

If a , b , and c are expanded about point j and the resulting differential equation is assumed to apply in the interval $[j-1, j+1]$, an exact solution of the resulting constant coefficient differential equation will be given as

$$F = A e^{a_j \eta} + B e^{b_j \eta} + \frac{c_j}{a_j b_j} \quad (45)$$

Using this solution to evaluate F_{j-1} , F_j , and F_{j+1} one obtains three equations that involve the two arbitrary constants A and B which can be eliminated to give the following difference equation:

$$\begin{aligned} F_{j+1} &= (e^{a_j \Delta \eta} + e^{b_j \Delta \eta}) F_j + e^{a_j \Delta \eta} e^{b_j \Delta \eta} F_{j-1} \\ &= \frac{c_j}{a_j b_j} \{ 1 - (e^{a_j \Delta \eta} + e^{b_j \Delta \eta}) + e^{a_j \Delta \eta} e^{b_j \Delta \eta} \} \end{aligned} \quad (46)$$

ORIGINAL PAGE IS
OF POOR QUALITY

A Taylor's series expansion of F_{j+1} and F_{j-1} about point j and the expansions of the exponentials involved for small $\Delta\eta$ shows that this difference equation is a second order accurate representation of equation (44) „ (see Appendix C).

For the special case when $b = c = 0$ and $a = C$, equation (44) reduces to equation (4b) and the difference equation (46) reduces to

$$F_{j+1} - (1 + e^{C\Delta\eta}) F_j + e^{C\Delta\eta} F_{j-1} = 0 \quad (47)$$

which has the exact characteristic roots

$$\lambda_1 = 1$$

and $\lambda_2 = e^{C\Delta\eta}$

Roscoe (Ref.10) used equation (47) to derive a "unified difference representation" in the form

$$\frac{C}{\Delta\eta(e^{C\Delta\eta} - 1)} [F_{j+1} - (1 + e^{C\Delta\eta})F_j + e^{C\Delta\eta} F_{j-1}] \quad (48)$$

to represent the differential expression $F_{\eta\eta} - CF_{\eta}$ with second order accuracy (here termed the RUDR).

If the exponentials in expression (48) are expanded and first order terms in $\Delta\eta$ are neglected this expression reduces to the first order upwind expression for positive C . The second order upwind expression follows if instead second order terms in $\Delta\eta$ are neglected. To get the corresponding expressions for negative C one first multiplies both the numerator and the denominator of (48) by $e^{-C\Delta\eta}$ and then expands the exponentials.

The central difference expression follows if the exponentials resulting after multiplying and dividing by $e^{-\Delta\eta C/2}$ are expanded to second order.

Instead of using (48) to represent $F_{\eta\eta} - C F_{\eta}$, one can use equation (46) which gives a consistent representation of the whole differential equation at the expense of the complications of evaluating the auxiliary roots a and b .

Consider now the limiting case when $a \rightarrow +\infty$ with b and C/a fixed. Equation (44) reduces to

$$-a F_{\eta} + abF = c \quad (49)$$

which corresponds to an inviscid flow of velocity a . In this case, equation (46) becomes

$$-F_j + e^{b_j \Delta\eta} F_{j-1} = \frac{c_j}{a_j b_j} (e^{b_j \Delta\eta} - 1) \quad (50)$$

which also follows if the approach used to derive equation (46) from equation (44) is applied to equation (49) in the interval $[j-1; j]$ with the coefficients a , b , and c evaluated at point j . This scheme is first order accurate (see Appendix C). An equation that involves F_j and F_{j+1} similar to equation (50) follows if the limit as $a \rightarrow -\infty$ is considered instead. Thus if this scheme is applied to a problem where there is a transition from viscous region to an inviscid one its accuracy will change from second order to first.

3. The Exponential Box Scheme

To overcome this difficulty let the coefficients in equation (44) be evaluated at the midpoint "m" and the resulting differential equation be valid in the interval $[j-1, j]$. Then the exact solution can be obtained in the form

$$F = A_1 e^{a_1 \eta} + B_1 e^{b_1 \eta} + \frac{c_1}{a_1 b_1} \quad (51)$$

where the subscript "1" refers to the interval $[j-1, j]$.

With $\eta=0$ and $\eta=\Delta\eta$ at the $(j-1)$ th and j th points respectively one gets the following expressions for F_{j-1} , F_j and $F_{\eta j}$

$$F_{j-1} = A_1 + B_1 + \frac{c_1}{a_1 b_1} \quad (52)$$

$$F_j = A_1 e^{a_1 \Delta\eta} + B_1 e^{b_1 \Delta\eta} + \frac{c_1}{a_1 b_1} \quad (53)$$

$$F_{\eta j} = a_1 A_1 e^{a_1 \Delta\eta} + b_1 B_1 e^{b_1 \Delta\eta} \quad (54)$$

Three other equations for the interval $[j, j+1]$ can be obtained in the form

$$F_j = A_2 + B_2 + \frac{c_2}{a_2 b_2} \quad (55)$$

$$F_{j+1} = A_2 e^{a_2 \Delta\eta} + B_2 e^{b_2 \Delta\eta} + \frac{c_2}{a_2 b_2} \quad (56)$$

$$F_{\eta j} = a_2 A_2 + b_2 B_2 \quad (57)$$

where now the subscript "2" refers to the interval $[j, j+1]$ and a_2, b_2 , and c_2 are the values of a, b , and c at the midpoint "n". The requirement that F_{n_j} of equations (54) and (57) should be the same together with the other four equations make it possible to eliminate the four arbitrary constants A_1, B_1, A_2 and B_2 to get the difference equation

$$\begin{aligned}
& \frac{(a_2-b_2)e^{-a_2\Delta\eta}}{1-e^{-a_2\Delta\eta}e^{b_2\Delta\eta}} F_{j+1} - \left\{ \frac{a_2e^{-a_2\Delta\eta}e^{b_2\Delta\eta}-b_2}{1-e^{-a_2\Delta\eta}e^{b_2\Delta\eta}} + \frac{a_1-b_1e^{-a_1\Delta\eta}e^{b_1\Delta\eta}}{1-e^{-a_1\Delta\eta}e^{b_1\Delta\eta}} \right\} F_j \\
& + \frac{(a_1-b_1)e^{b_1\Delta\eta}}{1-e^{-a_1\Delta\eta}e^{b_1\Delta\eta}} F_{j-1} = \frac{(a_2-b_2)e^{-a_2\Delta\eta}}{1-e^{-a_2\Delta\eta}e^{b_2\Delta\eta}} \frac{c_2}{a_2b_2} \\
& - \left\{ \frac{a_2e^{-a_2\Delta\eta}e^{b_2\Delta\eta}-b_2}{1-e^{-a_2\Delta\eta}e^{b_2\Delta\eta}} \frac{c_2}{a_2b_2} + \frac{a_1-b_1e^{-a_1\Delta\eta}e^{b_1\Delta\eta}}{1-e^{-a_1\Delta\eta}e^{b_1\Delta\eta}} \frac{c_1}{a_1b_1} \right\} \\
& + \frac{(a_1-b_1)e^{b_1\Delta\eta}}{1-e^{-a_1\Delta\eta}e^{b_1\Delta\eta}} \frac{c_1}{a_1b_1} \quad (58)
\end{aligned}$$

If the limiting case of equation (49) is considered, equation (58) reduces to

$$-F_j + e^{b_1\Delta\eta} F_{j-1} = \frac{c_1}{a_1b_1} (e^{b_1\Delta\eta} - 1) \quad \text{for } a \rightarrow +\infty \quad (59a)$$

$$\text{and } -F_{j+1} + e^{b_2\Delta\eta} F_j = \frac{c_2}{a_2b_2} (e^{b_2\Delta\eta} - 1) \quad \text{for } a \rightarrow -\infty \quad (59b)$$

which are second order accurate (see Appendix C).

ORIGINAL PAGE IS
OF POOR QUALITY

Thus the scheme represented by equation (58) is second order accurate in both the inviscid and viscous regions and it switches smoothly from one region to the other following the upwind rules as seen from equations (59a and b).

In addition to this advantage this scheme has, like the previous one, the advantages that it is diagonally dominant for negative ab (which corresponds to favorable pressure gradient) and that the error terms are independent of the velocity ($a+b$) compared to the other finite difference schemes described previously. In fact the error is due to approximating the coefficients a , b , and c by constant values and hence it depends on how those coefficients behave with η .

V. APPLICATION TO FALKNER SKAN FLOWS WITH INJECTION

1. General Statement

Numerical solutions to Falkner-Skan equations with injection boundary conditions given in equations (1) and (2) were obtained using the central difference, corrected central difference, midpoint upwind, Roscoe's UDR, the three point exponential, and the exponential box-schemes.

The boundary condition (2b) was applied at a sufficiently large but finite distance η_e . A grid of uniform mesh $\Delta\eta$ and $N+1$ grid points ($\eta_e = N \Delta\eta$) was used to divide the region between the surface ($\eta = 0$) and the boundary layer edge ($\eta = \eta_e$) into N intervals.

Equations (1) were solved in an uncoupled sense. The momentum equation (1b) was linearized by evaluating the coefficients of F_η and F using the previous iteration values to give

$$F_{\eta\eta} - \bar{V} F_\eta - \beta \bar{F} F_\eta = -\beta \quad (60)$$

where the "barred" quantities are the previous iteration values.

Each numerical scheme was used to reduce equation (60) to a difference equation of the general form

$$\alpha_{1j} F_{j+1} + \alpha_{2j} F_j + \alpha_{3j} F_{j-1} = \alpha_{4j} \quad (61)$$

for $j = 2, 3, \dots, N$. The resulting $N-1$ equations together with the two boundary conditions

$$F_1 = 0 \quad \text{and} \quad F_{N+1} = 1 \quad (62)$$

form a tridiagonal set of $N+1$ equations that were solved with the standard Thomas algorithm to give F_j , $j=1, \dots, N+1$. Then the continuity equation (1a) with the boundary condition (2a) was solved for V_j , $j=1, \dots, N+1$. For the first five of the schemes considered (i.e. central, corrected central, windward, Roscoe's UDR, TP exponential schemes) the continuity equation was integrated by the trapezoidal rule which results in the difference equation

$$V_j = V_{j-1} - \frac{\Delta n}{2} (F_j + F_{j-1}) \quad (63)$$

For the exponential box scheme the expression for F in each interval was introduced into the continuity equation which was integrated analytically to obtain a difference equation for V .

This procedure was repeatedly applied to relax a guessed solution to a converged one. The initial guessed solution that was used in all cases was

$$F_1 = 0 \quad (64)$$

$$F_j = 1 \quad j = 2, \dots, N+1 \quad (65)$$

and the corresponding V_j 's were obtained by integration of equation (1a) with the trapezoidal rule as expressed in equation (63). The solution process was considered to be converged when the maximum difference between any two corresponding F 's in two consecutive iterations was less than 10^{-5} .*

* An additional condition on V is used. The maximum difference between any two corresponding V 's is normalized w.r.t. the old value of V if it is greater than unity and w.r.t. unity otherwise, and this normalized difference is required to be less than 10^{-5} . This condition assures that V has converged to the fifth decimal place if it is less than 1 and to five significant digits otherwise.

2. Difference Equations

(a) The Central Difference Scheme

Introducing the central difference expressions (5) and (6) for $F_{\eta j}$ and $F_{\eta\eta j}$ into equation (60) evaluated at point j straightforwardly gives the central difference equation in the form

$$(1 - \frac{1}{2} \Delta\eta \bar{V}_j) F_{j+1} - (2 + \Delta\eta^2 \beta \bar{F}_j) F_j + (1 + \frac{1}{2} \Delta\eta \bar{V}_j) F_{j-1} = -\Delta\eta^2 \beta \quad (66)$$

(b) The Corrected Central Difference Scheme

The differential expression $F_{\eta\eta j} - \bar{V}_j F_{\eta j}$ in equation (60) evaluated at point j is replaced by the difference expression

$$(1 + \frac{1}{12} \Delta\eta^2 \bar{V}_j^2) \frac{F_{j+1} - 2F_j + F_{j-1}}{\Delta\eta^2} - \bar{V}_j \frac{F_{j+1} - F_{j-1}}{2\Delta\eta}$$

to give the difference equation

$$\begin{aligned} (1 - \frac{1}{2} \Delta\eta \bar{V}_j + \frac{1}{12} \Delta\eta^2 \bar{V}_j^2) F_{j+1} - (2 + \frac{1}{6} \Delta\eta^2 \bar{V}_j^2 + \Delta\eta^2 \beta \bar{F}_j) F_j \\ + (1 + \frac{1}{2} \Delta\eta \bar{V}_j + \frac{1}{12} \Delta\eta^2 \bar{V}_j^2) F_{j-1} = -\Delta\eta^2 \beta \end{aligned} \quad (67)$$

(c) The Midpoint Upwind Scheme

In this scheme the difference equations are written at $N-1$ midpoints only. The midpoint at which V first becomes negative is not included. If at the midpoint "m" (Fig. 2) $\bar{V}_m$ is positive, equation (60) will be evaluated at "m" to give

$$F_{\eta\eta m} - \bar{V}_m F_{\eta m} - \beta \bar{F}_m F_m = -\beta \quad (68)$$

F_m and $F_{\eta m}$ are represented by

$$F_m = \frac{F_j + F_{j-1}}{2} + O(\Delta\eta^2) \quad (69a)$$

$$\text{and } F_{\eta m} = \frac{F_j - F_{j-1}}{\Delta\eta} + O(\Delta\eta^2) \quad (69b)$$

To find the representation of $F_{\eta\eta m}$ one uses the relation

$$F_{\eta\eta m} = F_{\eta\eta j} - \frac{1}{2} \Delta\eta F_{\eta\eta\eta m} + O(\Delta\eta^2) \quad (70a)$$

with $F_{\eta\eta\eta m}$ given by the derivative of equation (60) evaluated at "m" as

$$F_{\eta\eta\eta m} = V_m F_{\eta\eta m} + V_{\eta m} F_{\eta m} + 2\beta F_m F_{\eta m}$$

Using the continuity equation this becomes

$$F_{\eta\eta\eta m} = V_m F_{\eta\eta m} - (1 - 2\beta) F_m F_{\eta m}$$

Evaluating F_m and V_m from the previous iteration values and substituting into (70a) one gets

$$(1 + \frac{1}{2} \Delta\eta \bar{V}_m) F_{\eta\eta m} = F_{\eta\eta j} + \frac{1}{2} \Delta\eta (1-2\beta) \bar{F}_m F_{\eta m} + O(\Delta\eta^2) \quad (70b)$$

With a central difference representation of $F_{\eta\eta j}$ combined with equation (69a) for $F_{\eta m}$, this becomes

$$\begin{aligned} (1 + \frac{1}{2} \Delta\eta \bar{V}_m) F_{\eta\eta m} &= \frac{F_{j+1} - 2F_j + F_{j-1}}{\Delta\eta^2} + \frac{1}{2} \Delta\eta (1-2\beta) \bar{F}_m \frac{F_j - F_{j-1}}{\Delta\eta} \\ &+ O(\Delta\eta^2) \end{aligned} \quad (70c)$$

Substituting (69a), (69b) and (70c) into (68) the following difference equation is obtained

$$\begin{aligned}
 F_{j+1} &= [2 + \Delta\eta \bar{V}_m + \frac{1}{2} \Delta\eta^2 \bar{V}_m^2 - \frac{1}{2} \Delta\eta^2 (1-2\beta) \bar{F}_m + \Delta\eta^2 (1 + \frac{1}{2} \Delta\eta \bar{V}_m) \beta \bar{F}_m] F_j \\
 &+ [1 + \Delta\eta \bar{V}_m + \frac{1}{2} \Delta\eta^2 \bar{V}_m^2 - \frac{1}{2} \Delta\eta^2 (1-2\beta) \bar{F}_m - \Delta\eta^2 (1 + \frac{1}{2} \Delta\eta \bar{V}_m) \beta \bar{F}_m] F_{j-1} \\
 &= -\Delta\eta^2 \beta (1 + \frac{1}{2} \Delta\eta \bar{V}_m) \quad (71)
 \end{aligned}$$

When $\bar{V}_m$ is negative, equation (60) is evaluated at "n", the same procedure outlined above is applied and an equation similar to (71) is obtained.

In equation (71), $\bar{F}_m$ is taken to be the average of $\bar{F}_j$ and $\bar{F}_{j-1}$ with second order error. A similar treatment of $\bar{V}_m$ was found to cause large errors so that a fourth order accurate value of $\bar{V}_m$ was used. Using Taylor's series expansion, one can write

$$\bar{V}_m = \frac{\bar{V}_j + \bar{V}_{j-1}}{2} - \frac{1}{8} \Delta\eta^2 \bar{V}_{\eta\eta_m} + O(\Delta\eta^4)$$

The differentiation of the continuity equation gives

$$\bar{V}_{\eta\eta_m} = -\bar{F}_{\eta_m}$$

so that

$$\bar{V}_m = \frac{\bar{V}_j + \bar{V}_{j-1}}{2} + \frac{1}{8} \Delta\eta^2 \frac{\bar{F}_j - \bar{F}_{j-1}}{\Delta\eta} + O(\Delta\eta^4) \quad (72)$$

where the difference representation in (69b) is applied to

$$\bar{F}_{\eta_m}$$

(d) Roscoe's UDR Scheme

The difference expression (48) with C replaced by $\bar{V}_j$ is used to replace the differential expression $F_{nnj} - \bar{V}_j F_{nj}$ in equation (60) evaluated at the j th point. The resulting difference equation will be

$$\frac{\bar{V}_j}{\Delta \eta (e^{\bar{V}_j \Delta \eta} - 1)} [F_{j+1} - (1 + e^{\bar{V}_j \Delta \eta}) F_j + e^{\bar{V}_j \Delta \eta} F_{j-1}] - \beta \bar{F}_j F_j = -\beta \quad (73)$$

(e) Three Point Exponential Scheme

The coefficients in equation (60) are evaluated at the j th point so that

$$F_{nn} - \bar{V}_j F_{nn} - \beta \bar{F}_j F = -\beta \quad (74a)$$

which now corresponds to the general form given in equation (44). The auxiliary roots, a_j and b_j , are given by

$$a_j = \bar{V}_j/2 + \sqrt{(\bar{V}_j/2)^2 + \beta \bar{F}_j} \quad (74b)$$

$$\text{and } b_j = \bar{V}_j/2 - \sqrt{(\bar{V}_j/2)^2 + \beta \bar{F}_j} \quad (74c)$$

with $c_j = -\beta$.

The resulting difference equation is then given by equation (46).

Considerations must be taken to avoid dividing by zero in the right hand side of equation (46) when one or both of a_j and b_j are identically zero. To avoid this difficulty, the R.H.S. of equation (46) can be written as

$$\text{R.H.S.} = \frac{1 - e^{a_j \Delta \eta}}{a_j} \frac{1 - e^{b_j \Delta \eta}}{b_j} c_j \quad (75)$$

which has the obvious limit for $b_j \rightarrow 0$

$$\lim_{b_j \rightarrow 0} \frac{1 - e^{b_j \Delta \eta}}{b_j} = -\Delta \eta \quad (76)$$

so that

$$\text{R.H.S.} = -\Delta \eta \frac{1 - e^{a_j \Delta \eta}}{a_j} c_j \quad (77)$$

If both a_j and b_j are zeros then a limiting process on a_j similar to that in (76) puts (77) in the form

$$\text{R.H.S.} = \Delta \eta^2 c_j \quad (78)$$

It should be noted that this particular limit treatment results in a difference equation which corresponds exactly to the limit form of the differential equation under consideration. For example, if $b_j = 0$, the difference equation will be

$$F_{j+1} - (e^{a_j \Delta \eta} + 1)F_j + e^{a_j \Delta \eta} F_{j-1} = -\Delta \eta \frac{1 - e^{a_j \Delta \eta}}{a_j} c_j \quad (79)$$

which will be obtained if the exponential approach is applied to the differential equation

$$F_{\eta\eta} - a_j F_\eta = c_j \quad (80)$$

(f) Exponential Box Scheme

In the difference equation (58) the roots a_1 and b_1 are the auxiliary roots of the differential equation (60) given in equations (74b&c) with the j subscript being replaced by "m" of Figure 2. Likewise the a_2 and b_2 are the roots corresponding to the midpoint "n". For the present application, also $c_1 = c_2 = -\beta$.

Again, special treatment is necessary for cases when any of these auxiliary roots vanish. For example, the coefficient of c_2 in the right hand side of equation (58) can be put in the form

$$\frac{1}{1 - \bar{e}^{a_2 \Delta \eta} e^{b_2 \Delta \eta}} [\bar{e}^{a_2 \Delta \eta} \frac{1 - e^{b_2 \Delta \eta}}{b_2} + \frac{1 - \bar{e}^{a_2 \Delta \eta}}{a_2}] \quad (81)$$

and a limiting process similar to that in (76) can be applied if necessary with $1/1 - \bar{e}^{a_1 \Delta \eta} e^{b_1 \Delta \eta}$ dropped from both sides of equation (58) if all a_i 's and b_i 's are zeros ($i=1,2$).

As mentioned before, the integration of the continuity equation (1a) was performed in a way consistent with the treatment of the momentum equation. The expression for F in an interval $[j-1, j]$ given by (51) is substituted into (1a) which is then integrated to give

$$V = - \frac{A_1}{a_1} e^{a_1 \eta} - \frac{B_1}{b_1} e^{b_1 \eta} - \frac{c_1}{a_1 b_1} \eta + D_1 \quad (82)$$

where A_1 and B_1 are obtained in terms of F_{j-1} and F_j from equations (52) and (53). Evaluating (82) at the $j-1^{\text{th}}$ and j^{th} points where $\eta=0$ and $\Delta \eta$ respectively and then subtracting, a difference equation for V is obtained in the form

$$V_j = V_{j-1} + \frac{A_1}{a_1} (1 - e^{a_1 \Delta \eta}) + \frac{B_1}{b_1} (1 - e^{b_1 \Delta \eta}) - \frac{c_1}{a_1 b_1} \Delta \eta \quad (83)$$

To evaluate the midpoint values of F and V which are necessary for evaluating the auxiliary roots a_i and b_i , the expressions in (51) and (82) are used.

To avoid the difficulties that arise due to this need of evaluating F and V at the midpoints another version of the EB scheme is obtained and described in Appendix D. Also, a new inversion algorithm that can be used with that scheme is presented in that Appendix.

ORIGINAL PAGE IS
OF POOR QUALITY

VI. NUMERICAL RESULTS AND DISCUSSION

1. General Statement

Three cases of the Falkner-Skan equations with different values of the pressure gradient coefficient β and the injection velocity V_w were solved over a range of mesh step sizes. A step size study for the calculated velocity F at chosen points was performed to answer two questions:

1. How long would each scheme considered stay second order accurate as the step size $\Delta\eta$ increases?, and
2. How accurate is each scheme compared to the others in the different problems considered?

In addition, attention was focused on the numerical behavior of the velocity profiles to determine if local oscillatory behavior ("wiggles") was encountered.

In all cases, the calculated F at the chosen points was plotted versus the square of the step size and the resulting curves extrapolated to zero step size to provide an "exact" value as a basis of comparison. In general, these plots should be straight lines for small step sizes since the schemes considered were all second order accurate in $\Delta\eta$. The answer to the first question above can therefore be determined by observing how long these plots remain straight as the step size was increased, while the answer to the second question is obtained by comparing the slopes of these terminal line segments.

2. Case I - The Blasius Equations

As a standard of comparison, attention is first directed to the familiar flat plate case corresponding to zero pressure gradient ($\beta = 0$) and no injection ($V_w = 0$). The momentum equation (1b) reduces to

$$F_{\eta\eta} - VF_{\eta} = 0 \quad (84)$$

The F-velocity profile is shown in Figure 4, where it is seen that the thin boundary layer is confined to the region $\eta < 6$. Figure 5 shows the two individual terms of equation (84) demonstrating the complete balance of these terms at all points across the viscous layer.

The two points $\eta = 1$ and $\eta = 2$ were chosen to perform the step size study. The first point is a typical point in the boundary layer, while the second is the point where the diffusion and convection effects have maximum values (see Figure 5).

In Figures 6 and 7, four step sizes $\Delta\eta = 0.2, 0.25, 0.3,$ and 0.5 , are used to plot F at $\eta = 1$ with straight segments joining the points so that any change in the slope can be observed. In Figure 6, a straight line is obtained by the central difference scheme (C), while the upwind Scheme (UW) shows deviation from the straight line as the step size increases. The corrected central difference scheme (CC) which is seen to be more accurate than the other two, shows a slight curvature. In Figure 7, the three point exponential scheme (TPE) and the Roscoe's UDR scheme

(RUDR) which are identical for this special problem give a linear curve over the chosen range of $\Delta\eta$. This curve is very close to that of the CC-scheme and in fact represents a straightened version of the scheme. This observation (for the RUDR scheme only) is found to be true in all the results obtained in this study. The exponential box scheme (EB) is also seen to follow a straight line over this range of $\Delta\eta$ and is slightly less accurate than the RUDR and TPE schemes.

In Figures 8 and 9, one more step size $\Delta\eta = 0.4$ is used to plot F at $\eta = 2$. Again, in Figure 8 the CC-scheme shows better accuracy than both the C-scheme (which also loses linearity) and the UW-scheme (which displays a subtle nonsystematic behavior in that its slope increases and decreases from one segment to another). These behaviors of the C and UW-schemes are presumed to be present for $F(1)$ as shown in Figure 6, although they were not observed within the accuracy of the plots. As shown in Figure 9, all the exponential schemes produce straight line variations over the entire range of $\Delta\eta$ for this point.

In Tables 1 and 2, the detailed velocity profiles are presented for a step size of 0.5, where local oscillations are observed in the central difference calculations near the outer edge when $|\dot{V}| > 4$, i.e. when $\Delta\eta |V| > 2$ which is in accordance with condition (16).

3. Case II - Moderate Injection at an Axisymmetric Stagnation Point

Here an injection velocity of $V_w = 4$ is used with a pressure gradient coefficient of $\beta = 0.5$, so that a region of primarily inviscid flow near the surface is created. In this inviscid region ($\eta < 5$), the velocity profile is shown in Figure 10 to be nearly linear. Above this region ($\eta > 5$), the diffusion term begins to play an equal role in the force balance represented by the momentum equation. This classification is more clear in Figure 11 which presents the individual contributions of the diffusion, convection, and pressure gradient terms in the momentum equation. In the step size study conducted here, 3 points in the layer were considered, namely the points: $\eta = 6$ which lies in the bottom of the viscous region, $\eta = 8$ near which the convection effects are negligible and the normal velocity V changes its sign, and $\eta = 10$ where all the terms effects are vanishingly small but in perfect balance. Step sizes of 0.4, 0.5, 0.6667 and 1.0 are used to plot F at those three points.

In Figures 12 and 13, the basically inviscid values of $F(6)$ are presented. The three curves in Figure 12 are not straight, with the CC-scheme being most nearly linear and, moreover, more accurate than the C-scheme. The UW-scheme gives better accuracy than the other two schemes, but is seen to be unreliable over this region of $\Delta\eta$. In Figure 13, both the RUDR and EB-schemes have very slight curvature with the RUDR scheme being the more accurate of the two. The TPE-scheme is found to give large errors and to lose its second order accuracy even with step size of 0.4. Figures 14 and 15 show the velocity at a viscous point, $F(8)$ near the point where V changes its sign. Thus large

errors should be expected to be associated with the UW-scheme due to its required direction switching. This is the result observed in Figure 14 where again the CC-scheme is seen to be the more accurate of the finite difference schemes.

The exponential scheme results at the same viscous point $\eta = 8$ are shown in Figure 15, in which it is seen that the EB-scheme has become more accurate than the others and that the TPE-scheme encounters obvious curvature. Observations similar to those of Figure 12 for $F(6)$ are obtained from Figure 16 for $F(10)$. On the other hand, Figure 17 for the exponential scheme results for $F(10)$ resembles those of Figure 15 for $F(8)$.

The F -velocity profiles for this case are presented in Tables 3 and 4 for step size of 1.0. For this moderate blowing case, oscillations are observed in the central difference calculations, not only near the outer edge as before, but also in the basically inviscid region close to the surface where again oscillatory roots to the difference equations exist. Errors are observed in the calculations by the totally exponential schemes (TPE and EB-schemes) in the region close to the surface where the velocity profile is linear. These errors are larger in the case of the TPE scheme.

To this point, the following conclusions can be made based on the moderate blowing results presented here:

1. The CC-scheme is more accurate than the C-scheme.
2. Apart from the nonsystematic behavior of the UW-scheme due to the errors introduced at the region in which V changes its

sign, the UW-scheme gives reasonably accurate results.

3. The RUDR-scheme represents a straightening of the CC-scheme.

4. The RUDR and EB schemes are the two schemes that have the slightest deviation from the second order accuracy at large step sizes. The EB scheme shows better accuracy as the injection velocity is increased, which should be expected since the transition from the inviscid region created by the injection to the viscous region is better honored in the EB-scheme.

5. The TPE-scheme is not comparable with the other exponential schemes, neither in its accuracy nor in its behavior with the step size. The errors observed in the velocity profile in the inviscid region close to the surface are blamed for that and are assumed to affect the solution everywhere else. One other possible reason for this limited accuracy could be that the integration of the continuity equation by the trapezoidal rule is not consistent with the remainder of the method.

6. The central difference solution is oscillatory whenever a condition similar to condition (16) is satisfied. Oscillations do not occur in any other scheme. However, obvious errors are observed when exponentials are used to approximate inviscid linear regions of the profiles.

4. Case III - Strong Blowing at an Axisymmetric Stagnation Point

To examine the validity of the previous conclusions, a high injection rate $V_w = 25$ is applied with the pressure gradient coefficient $\beta = 0.5$. As shown in Figures 18 and 19, a large region ($0 < \eta < 47$) of inviscid flow is formed in which the

velocity profile is essentially linear followed by a very small viscous region ($47 < \eta < 53$) and then a region ($\eta > 53$) in which the diffusion, convection, and pressure effects are negligible. The points considered for detailed analysis of the numerical results are $\eta = 48, 50$, and 52 for the same reasons discussed in the previous case and the step sizes used are $0.5, 0.8, 1.0$, and 2.0 . In Figure 20 the CC-scheme in the inviscid region shows a large turn and loss of its second order accurate nature for $\Delta\eta > 0.8$. The UW-scheme shows a nonsystematic behavior in that the slopes of its segments increase and decrease alternatively and the segment of smaller step sizes does not head towards the exact value of $F(48)$. Large non second order truncation errors are shown for the C-scheme.

In Figure 21, only the EB and the RUDR schemes appear because the TPE scheme showed very poor accuracy at that point in the profile. The EB and RUDR schemes give the same solution for the smaller step sizes and both deviate slightly from the second order result for $\Delta\eta > 1$.

Figure 22 shows the finite difference results at the switch point ($V = 0$ at $\eta \approx 50$), where it is seen that the UW-scheme gives again large truncation errors and nonsystematic behavior. The C and CC-schemes are seen to lose their second order accuracy for $\Delta\eta > 1$. In Figure 23, the EB-scheme, as expected, shows better accuracy than the RUDR, while the TPE-scheme follows its usual behavior observed before. These observations are further demonstrated for the edge region ($\eta = 52$) in Figures 24 and 25 where $F(52)$ is presented. Large truncation errors and

overshoot ($F > 1$) are encountered with the C-scheme, while the CC-scheme shows a dramatic loss in second order accuracy for large $\Delta\eta$. The UW-scheme is obviously nonsystematic; the TPE-scheme does not behave in a second order fashion; the RUDR scheme is fairly accurate and it represents a straightening of the CC-scheme; and the EB-scheme is the most accurate and consistently maintains second order accuracy even for the large step size of 2.0.

The velocity profiles presented in Tables 5 and 6 support the previous observations that the C-scheme is oscillatory in the inviscid region close to the surface and the region close to the outer edge whenever a condition similar to condition (16) is satisfied. Also, the errors observed in the linear region close to the surface in the TPE and EB calculations are observed again. These errors are due to approximating a straight line by exponential segments and can be reduced by refining the matching between these exponential segments and by considering the next term in the asymptotic expansions assumed for the coefficients a , b , and c in the differential equation (44).

ORIGINAL PAGE IS
OF POOR QUALITY

VII. CONCLUSION AND RECOMMENDATIONS

A study of numerical schemes for solving viscid/inviscid fluid flow problems has been presented. Truncation errors of the oscillatory type associated with the central difference solutions of these problems for large step sizes have been related to an explicit downstream contribution to the convection effects which violates the physical process of convection.

It has been found that a natural correction to the central difference scheme (which is equivalent to an addition of artificial viscosity like term) has improved the accuracy of the central difference scheme, increased its range of possible applicability, and eliminated the numerical "wiggles". A fourth order scheme obtained using this same line of thought has been recently presented by Wornom (Ref. 17).

A second order accurate upwind scheme has been described and found to suffer a severe conservation problem at the switch point although it has shown reasonable accuracy away from that point. It is recommended that this point should be studied further to solve this switching problem.

The exponential scheme proposed by Spalding and used by Roscoe has been interpreted in a way (see Pruess, Ref. 8) that has allowed a development of the exponential box scheme. This interpretation gives also a rational basis for introducing higher order corrections.

The numerical calculations carried out with the exponential box scheme have been found to be very promising. Further study

is recommended both in the application of the scheme to a wider range of problems and in considering the higher order corrections that can be introduced to it.

ORIGINAL PAGE IS
OF POOR QUALITY

VIII. REFERENCES

1. Roache, P.J., "Computational Fluid Dynamics," Hermosa Publishers, Albuquerque, NM, 1976, pp. 161-165.
2. Hirsh, R.S., Rudy, D.H., "The role of diagonal dominance and cell Reynolds number in implicit difference methods for fluid mechanics problems," J. Comp. Phy., Nov. 1974, pp. 304-310.
3. Liu, T.M., Chiu, H.H., "Fast and stable numerical method for boundary layer flow with massive blowing," AIAA J. 1976, pp. 114-116.
4. Briley, W.R., McDonald, H., "Implicit numerical method for the multidimensional compressible Navier-Stokes equations," Rept. M911363-6, United Aircraft Res. Labs., Nov. 1973.
5. Garrett, L.B., Smith, G.L., Perkins, J.N., "An implicit finite difference solution to the viscous shock layer including the effects of radiation and strong blowing," NASA TR R-388, Nov. 1972.
6. Raithby, G.D., "A Critical Evaluation of Upstream Differencing Applied to Problems Involving Fluid Flow," Comp. Methods in App. Mech's and Engrg., V. 9, 1976, pp. 75-103.
7. Spalding, D.B., "A novel finite difference formulation for differential expressions involving both first and second derivatives," Inter. J. for Numer. Methods in Engrg., V. 4, 1972, pp. 551-559.
8. Pruess, Steven A., "Solving Linear Boundary Value Problems by Approximating the Coefficients," Math of Comp., V. 27, 1973, pp. 551-561.
9. Rose, Milton E., "Weak-element Approximations to Elliptic Differential Equations," Numer. Math. V. 24, 1975, pp. 185-204.
10. Roscoe, D.F., "New methods for the derivation of stable difference representations for differential equations," J. Inst. Maths. Applics., V. 16, 1975, pp. 291-301.

11. Roscoe, D.F., "The solution of the three dimensional Navier-Stokes equations using a new finite-difference approach," Inter. Jour. for Numer. Methods in Engrg.," V. 10, 1976, pp. 1299-1308.
12. Chien, John C., "A General Finite-Difference Formulation with Application to Navier-Stokes Equations," J. of Comp. Phys., V. 20, 1976, pp. 268-278.
13. Varga, R.S., "Matrix Iterative Analysis," Printice Hall, Inc., Englewood Cliffs, New Jersey, 1962, pp. 23, 161-171.
14. Blottner, F.G., "Computational techniques for boundary layers," Agard Lecture Series No. 73, Feb. 1975, pp. 3-9.
15. Atias, M., Wolfshtein, M., Israeli, M., "Efficiency of Navier-Stokes solvers," AIAA J., V. 15, Feb. 1977, pp. 263-266.
16. Khosla, P.K., Rubin, S.G., "A diagonally dominant second order accurate implicit scheme," Computers and Fluids, 1974, pp. 207-209.
17. Wornom, S.F., "A fourth order box method for solving the boundary layer equations," NASA TM X-74000, Jan. 1977.
18. Keller, H.B., "Numerical Methods for Two-Point Boundary Value Problems," Blaisdel Publishing Co., 1968, pp. 72-75.

ORIGINAL PAGE IS
OF POOR QUALITY

APPENDIX A

WIGGLES AND DIAGONAL DOMINANCE

The numerical oscillations or "wiggles" associated with the central difference calculations for large step sizes have been attributed, incorrectly, to round off errors produced due to the lack of diagonal dominance in the difference equation. In this appendix, it will be shown that the wiggles and the lack of diagonal dominance are two different problems that can occur independently. Consider the difference equation (8) which will be rewritten here for convenience

$$\alpha_1 F_{j+1} + \alpha_2 F_j + \alpha_3 F_{j-1} = \alpha_4 \quad (A1)$$

where $\alpha_1 > 0$.

The characteristic equation corresponding to equation (A1) is

$$\alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3 = 0 \quad (A2)$$

whose roots are

$$\lambda_1 = \frac{-\alpha_2 + \sqrt{\alpha_2^2 - 4\alpha_1\alpha_3}}{2\alpha_1} \quad (A3)$$

and

$$\lambda_2 = \frac{-\alpha_2 - \sqrt{\alpha_2^2 - 4\alpha_1\alpha_3}}{2\alpha_1} \quad (A4)$$

Consider now the following cases:

Case I $\alpha_3 < 0$

In this case λ_2 is always negative while λ_1 is always positive.

Case II $\alpha_2^2 > 4\alpha_1\alpha_3 > 0$ and $\alpha_2 > 0$

Both λ_1 and λ_2 are negative in this case.

Case III $\alpha_2^2 > 4\alpha_1\alpha_3 > 0$ and $\alpha_2 < 0$

Both λ_1 and λ_2 are positive.

Case IV $\alpha_2^2 < 4\alpha_1\alpha_3$

Complex roots λ_1 and λ_2 are obtained.

Thus oscillations are not expected in Case III, while they are expected in Cases I and II. In Case IV, λ_1 and λ_2 can be put in the form $\rho e^{\pm i\theta}$ and the exact solution for the difference equation will have the form

$$F_j = \rho^j [R_1 \cos j\theta + R_2 \sin j\theta] \quad (A5)$$

Since the set of equations (A1) for $j=1, \dots, N+1$ has real coefficients, R_1 and R_2 will be such that all the F_j 's are real and the solution profile will have harmonic behavior.

Diagonal Dominance

Consider the system of equations (A1) for $j=1, \dots, N+1$. The diagonal dominance condition for this system is

$$|\alpha_2| \geq |\alpha_1| + |\alpha_3| \quad (A6)$$

for all j .

ORIGINAL PAGE IS
OF POOR QUALITY

This condition is the sufficient condition for the matrix of the coefficients to be nonsingular and hence invertible (Varga, Ref.13). Also, it has been shown by Keller (Ref. 18) that condition (A6) is a sufficient condition for the Thomas algorithm to be nonsingular.

It is clear from the previous analysis that diagonal dominance has to do with the magnitude of the coefficients α_1 , α_2 and α_3 , while the wiggles have to do with their signs. E.g., if $\alpha_3 < 0$ oscillations will take place according to Case I above, although diagonal dominance can still be satisfied.

However, in the case when $\alpha_2 < 0$ and $\alpha_3 > 0$, it can be shown that the diagonal dominance is a sufficient condition for the roots λ_1 and λ_2 to be real and, according to Case III, positive.

Squaring (A6) and noting that α_1 and α_3 are positive, one gets

$$\alpha_2^2 \geq \alpha_1^2 + \alpha_3^2 + 2\alpha_1\alpha_3$$

so that

$$\alpha_2^2 - 4\alpha_1\alpha_3 \geq (\alpha_1 - \alpha_3)^2 \geq 0$$

which proves that λ_1 and λ_2 are real.

APPENDIX B

A GENERAL THREE POINT FINITE DIFFERENCE EQUATION

The approach used to obtain the midpoint upwind scheme in Chapter II can be used to produce a family of finite difference schemes if it is applied to a general point "s" a distance $\alpha \Delta \eta$ below point j where $-1 \leq \alpha \leq +1$.

Consider equation (4b) and its derivative evaluated at point s. They give

$$F_{\eta\eta_s} - C F_{\eta_s} = 0 \quad (B1)$$

and
$$F_{\eta\eta\eta_s} - C F_{\eta\eta_s} = 0 \quad (B2)$$

For $0 \leq \alpha \leq 1$, a Taylor's series expansion of $F_{\eta\eta_j}$ gives

$$F_{\eta\eta_j} = F_{\eta\eta_s} + \alpha \Delta \eta F_{\eta\eta\eta_s} + O(\Delta \eta^2)$$

Substituting for $F_{\eta\eta\eta_s}$ from (B2) one gets

$$(1 + \alpha \Delta \eta C) F_{\eta\eta_s} = F_{\eta\eta_j} + O(\Delta \eta^2) \quad (B3)$$

Similarly, a Taylor's series expansion of F_{η_m} and the use of (B1) gives

$$[1 - (\frac{1}{2} - \alpha) \Delta \eta C] F_{\eta_s} = F_{\eta_m} + O(\Delta \eta^2) \quad (B4)$$

Using central differences to represent F_{η_m} and $F_{\eta\eta_j}$, the following second order accurate finite difference approximation of equation (4b) is obtained:

$$\begin{aligned}
[1 - \frac{1}{2}(1-2\alpha)\Delta\eta C]F_{j+1} - [2 + 2\alpha\Delta\eta C + \alpha\Delta\eta^2 C^2]F_j \\
+ [1 + \frac{1}{2}(1+2\alpha)\Delta\eta C + \alpha\Delta\eta^2 C^2]F_{j-1} = 0
\end{aligned} \quad (B5)$$

A similar expression can be obtained for the case where $-1 \leq \alpha \leq 0$ with F_{η_n} used instead of F_{η_m} . Combining both cases, one gets:

$$\begin{aligned}
[1 - \frac{1}{2}(1-2\alpha)\Delta\eta C - \frac{1}{2}(\alpha - |\alpha|)\Delta\eta^2 C^2]F_{j+1} \\
- [2 + 2\alpha\Delta\eta C + |\alpha|\Delta\eta^2 C^2]F_j \\
+ [1 + \frac{1}{2}(1+2\alpha)\Delta\eta C + \frac{1}{2}(\alpha + |\alpha|)\Delta\eta^2 C^2]F_{j-1} = 0
\end{aligned} \quad (B6)$$

for $-1 \leq \alpha \leq 1$.

The corresponding second order error is given by

$$\epsilon = -(\frac{1}{12} + \frac{1}{2}\alpha^2)\Delta\eta^2 F_{\eta\eta\eta\eta} + [\frac{1}{24} + \frac{1}{2}(\frac{1}{2} - |\alpha|)^2]\Delta\eta^2 C F_{\eta\eta\eta} \quad (B7)$$

where $F_{\eta\eta\eta}$ and $F_{\eta\eta\eta\eta}$ are evaluated at some point and can be related to each other through the differential equation (4b) so that

$$\epsilon = [-\frac{1}{12} + \frac{1}{2}\alpha^2] + \frac{1}{24} + \frac{1}{2}(\frac{1}{2} - |\alpha|)^2 \Delta\eta^2 F_{\eta\eta\eta\eta} \quad (B8)$$

Note that when $\alpha = 0$, equation (B5) reduces to the central difference equation (7), when $\alpha = \frac{1}{2}$ the branch of the midpoint upwind scheme corresponding to $C > 0$ is obtained and, when $\alpha = -\frac{1}{2}$ the branch corresponding to $C < 0$ results in.

APPENDIX C

ERROR ANALYSIS FOR EXPONENTIAL SCHEMES

1. Error Analysis of Equation (46)

Multiplying equation (46) by $e^{-a_j \Delta n/2} e^{-b_j \Delta n/2}$ results in the following equation

$$\begin{aligned} e^{-a_j \Delta n/2} e^{-b_j \Delta n/2} F_{j+1} - [e^{a_j \Delta n/2} e^{-b_j \Delta n/2} + e^{-a_j \Delta n/2} e^{b_j \Delta n/2}] F_j \\ + e^{a_j \Delta n/2} e^{b_j \Delta n/2} F_{j-1} = \frac{c_j}{a_j b_j} \{ e^{-a_j \Delta n/2} e^{-b_j \Delta n/2} \\ - [e^{a_j \Delta n/2} e^{-b_j \Delta n/2} + e^{-a_j \Delta n/2} e^{b_j \Delta n/2}] + e^{a_j \Delta n/2} e^{b_j \Delta n/2} \} \quad (C1) \end{aligned}$$

Expanding the exponentials for small Δn one gets

$$\begin{aligned} e^{-a_j \Delta n/2} e^{-b_j \Delta n/2} &= 1 - \frac{1}{2} \Delta n (a_j + b_j) + \frac{1}{8} \Delta n^2 (a_j + b_j)^2 - \frac{1}{48} \Delta n^3 (a_j + b_j)^3 + O(\Delta n^4) \\ e^{a_j \Delta n/2} e^{b_j \Delta n/2} &= 1 + \frac{1}{2} \Delta n (a_j + b_j) + \frac{1}{8} \Delta n^2 (a_j + b_j)^2 + \frac{1}{48} \Delta n^3 (a_j + b_j)^3 + O(\Delta n^4) \\ e^{-a_j \Delta n/2} e^{b_j \Delta n/2} &= 1 - \frac{1}{2} \Delta n (a_j - b_j) + \frac{1}{8} \Delta n^2 (a_j - b_j)^2 - \frac{1}{48} \Delta n^3 (a_j - b_j)^3 + O(\Delta n^4) \\ e^{a_j \Delta n/2} e^{-b_j \Delta n/2} &= 1 + \frac{1}{2} \Delta n (a_j - b_j) + \frac{1}{8} \Delta n^2 (a_j - b_j)^2 + \frac{1}{48} \Delta n^3 (a_j - b_j)^3 + O(\Delta n^4) \end{aligned} \quad (C2)$$

Substitution in the right hand side of equation (C1) gives

$$\text{R.H.S.} = \Delta n^2 C_j + O(\Delta n^4) \quad (C3)$$

Expanding F_{j+1} and F_{j-1} about point j one gets

$$F_{j+1} = F_j + \Delta n F_{nj} + \frac{1}{2} \Delta n^2 F_{nnj} + \frac{1}{6} \Delta n^3 F_{nnnj} + O(\Delta n^4)$$

ORIGINAL PAGE IS
OF POOR QUALITY

$$F_{j-1} = F_j - \Delta\eta F_{\eta j} + \frac{1}{2} \Delta\eta^2 F_{\eta\eta j} - \frac{1}{6} \Delta\eta^3 F_{\eta\eta\eta j} + O(\Delta\eta^4) \quad (C4)$$

Substitution in the left hand side of equation (C1) gives

$$\begin{aligned} \text{coeff. of } F_j &= 1 - \frac{1}{2} \Delta\eta (a_j + b_j) + \frac{1}{8} \Delta\eta^2 (a_j + b_j)^2 - \frac{1}{48} \Delta\eta^3 (a_j + b_j)^3 + O(\Delta\eta^4) \\ &\quad - 2 \quad - \frac{1}{4} \Delta\eta^2 (a_j - b_j)^2 + O(\Delta\eta^4) \\ &\quad + 1 + \frac{1}{2} \Delta\eta (a_j + b_j) + \frac{1}{8} \Delta\eta^2 (a_j + b_j)^2 + \frac{1}{48} \Delta\eta^3 (a_j + b_j)^3 + O(\Delta\eta^4) \\ &= \Delta\eta^2 a_j b_j + O(\Delta\eta^4) \end{aligned}$$

$$\begin{aligned} \text{coeff. of } F_{\eta j} &= \Delta\eta - \frac{1}{2} \Delta\eta^2 (a_j + b_j) + \frac{1}{8} \Delta\eta^3 (a_j + b_j)^2 + O(\Delta\eta^4) \\ &\quad - \Delta\eta - \frac{1}{2} \Delta\eta^2 (a_j + b_j) - \frac{1}{8} \Delta\eta^3 (a_j + b_j)^2 + O(\Delta\eta^4) \\ &= - \Delta\eta^2 (a_j + b_j) + O(\Delta\eta^4) \end{aligned}$$

$$\begin{aligned} \text{coeff. of } F_{\eta\eta j} &= \frac{1}{2} \Delta\eta^2 - \frac{1}{4} \Delta\eta^3 (a_j + b_j) + O(\Delta\eta^4) \\ &\quad + \frac{1}{2} \Delta\eta^2 + \frac{1}{4} \Delta\eta^3 (a_j + b_j) + O(\Delta\eta^4) \\ &= \Delta\eta^2 + O(\Delta\eta^4) \end{aligned}$$

$$\begin{aligned} \text{coeff. of } F_{\eta\eta\eta j} &= \frac{1}{6} \Delta\eta^3 + O(\Delta\eta^4) \\ &\quad - \frac{1}{6} \Delta\eta^3 + O(\Delta\eta^4) \\ &= O(\Delta\eta^4) \end{aligned}$$

so that

$$\text{L.H.S} = \Delta\eta^2 [F_{\eta\eta_j} - (a_j + b_j)F_{\eta_j} + a_j b_j F_j] + O(\Delta\eta^4) \quad (\text{C5})$$

Equating (C3) and (C5), one gets

$$F_{\eta\eta_j} - (a_j + b_j)F_{\eta_j} + a_j b_j F_j = c_j + O(\Delta\eta^2)$$

which shows that equation (46) is a second order accurate representation of equation (44).

2. Error Analysis of Equation (50)

Multiplying equation (50) by $e^{-b_j \Delta\eta/2}$ results in

$$-e^{-b_j \Delta\eta/2} F_j + e^{b_j \Delta\eta/2} F_{j-1} = \frac{c_j}{a_j b_j} (e^{b_j \Delta\eta/2} - e^{-b_j \Delta\eta/2}) \quad (\text{C7})$$

Expanding the exponentials in the right hand side, one gets

$$\begin{aligned} \text{R.H.S.} &= \frac{c_j}{a_j b_j} \left\{ \left[1 + \frac{1}{2} \Delta\eta b_j + \frac{1}{8} \Delta\eta^2 b_j^2 + O(\Delta\eta^3) \right] \right. \\ &\quad \left. - \left[1 - \frac{1}{2} \Delta\eta b_j + \frac{1}{8} \Delta\eta^2 b_j^2 + O(\Delta\eta^3) \right] \right\}, \\ &= \Delta\eta \frac{c_j}{a_j} + O(\Delta\eta^3) \end{aligned} \quad (\text{C8})$$

Expanding the exponentials in the left hand side and F_{j-1} about point j one gets

ORIGINAL PAGE IS
OF POOR QUALITY

$$\begin{aligned}
\text{L.H.S.} &= -[1 - \frac{1}{2} \Delta \eta b_j + \frac{1}{8} \Delta \eta^2 b_j^2 + O(\Delta \eta^3)] F_j \\
&\quad + [1 + \frac{1}{2} \Delta \eta b_j + \frac{1}{8} \Delta \eta^2 b_j^2 + O(\Delta \eta^3)] [F_j - \Delta \eta F_{\eta j} + \frac{1}{2} \Delta \eta^2 F_{\eta \eta j} + O(\Delta \eta^3)] \\
&= \Delta \eta b_j F_j - \Delta \eta (1 + \frac{1}{2} \Delta \eta b_j) F_{\eta j} + \frac{1}{2} \Delta \eta^2 F_{\eta \eta j} + O(\Delta \eta^3) \quad (C9)
\end{aligned}$$

So that, upon equating (C8) and (C9) there follows:

$$-F_{\eta j} + b_j F_j = \frac{c_j}{a_j} - \frac{1}{2} \Delta \eta (F_{\eta \eta j} - b_j F_{\eta j}) + O(\Delta \eta^2) \quad (C10)$$

which shows that equation (50) is a first order accurate representation of equation (49). It is second order accurate only if the coefficients a , b , and c are constants.

3. Error Analysis of Equation (58)

An error analysis for equation (58) has been carried out by expanding its coefficients and reducing it to the three point difference equation obtained by applying the Keller's box scheme to the same differential equation (44). The difference between the two schemes is found to be $O(\Delta \eta^2)$ which shows that the EB scheme is second order accurate since the Keller's box scheme is known to be second order accurate.

4. Error Analysis of Equation (59a)

Multiplying by $e^{-b_1 \Delta \eta / 2}$ one gets

$$-e^{-b_1 \Delta \eta / 2} F_j + e^{b_1 \Delta \eta / 2} F_{j-1} = \frac{c_1}{a_1 b_1} [e^{b_1 \Delta \eta / 2} - e^{-b_1 \Delta \eta / 2}] \quad (C11)$$

$$\begin{aligned}
\text{R.H.S.} &= \frac{1}{a_1 b_1} \left\{ \left[1 + \frac{1}{2} \Delta \eta b_1 + \frac{1}{8} \Delta \eta^2 b_1^2 + O(\Delta \eta^3) \right] \right. \\
&\quad \left. - \left[1 - \frac{1}{2} \Delta \eta b_1 + \frac{1}{8} \Delta \eta^2 b_1^2 + O(\Delta \eta^3) \right] \right\} \\
&= \Delta \eta \frac{c_1}{a_1} + O(\Delta \eta^3)
\end{aligned} \tag{C12}$$

$$\begin{aligned}
\text{L.H.S.} &= - \left[1 - \frac{1}{2} \Delta \eta b_1 + \frac{1}{8} \Delta \eta^2 b_1^2 + O(\Delta \eta^3) \right] \left[F_m + \frac{1}{2} \Delta \eta F_{\eta_m} + \frac{1}{8} \Delta \eta^2 F_{\eta \eta_m} + O(\Delta \eta^3) \right] \\
&\quad + \left[1 + \frac{1}{2} \Delta \eta b_1 + \frac{1}{8} \Delta \eta^2 b_1^2 + O(\Delta \eta^3) \right] \left[F_m - \frac{1}{2} \Delta \eta F_{\eta_m} + \frac{1}{8} \Delta \eta^2 F_{\eta \eta_m} + O(\Delta \eta^3) \right] \\
&= \Delta \eta b_1 F_m - \Delta \eta F_{\eta_m} + O(\Delta \eta^3)
\end{aligned} \tag{C13}$$

Equating (C12) and (C13), one gets

$$-F_{\eta_m} + b_1 F_m = \frac{c_1}{a_1} + O(\Delta \eta^2) \tag{C14}$$

which shows that equation (59a) is a second order accurate representation of equation (49).

ORIGINAL PAGE IS
OF POOR QUALITY

APPENDIX D

A SECOND GENERATION EXPONENTIAL BOX SCHEME

Let the coefficients a , b , and c in equation (44) be approximated to first order by their values at the grid point j and the solution (45) be valid in the interval between the two midpoints m and n . For convenience, equation (45) will be rewritten in the form

$$F = A_2 e^{a_2 n} + B_2 e^{b_2 n} + f_2 \quad (D1)$$

where

$$f_2 = \frac{c_2}{a_2 b_2} \quad (D2)$$

and the subscript "2" refers to point j .

Similar expressions will be valid at midpoint intervals below and above the previous interval with the coefficients a , b , and c approximated by their values at points $j-1$ and $j+1$, respectively.

Equation (D1) can be used to evaluate F_j , F_m , F_n , F_{n_m} and F_{n_n} in the forms

$$F_j = A_2 + B_2 + f_2 \quad (D3)$$

$$F_m = A_2 r_2^{-1} + B_2 t_2^{-1} + f_2 \quad (D4)$$

$$F_n = A_2 r_2^{-1} + B_2 t_2 + f_2 \quad (D5)$$

$$F_{n_m} = a_2 A_2 r_2 + b_2 B_2 t_2^{-1} \quad (D6)$$

$$F_{n_n} = a_2 A_2 r_2^{-1} + b_2 B_2 t_2 \quad (D7)$$

$$\text{where } r_2 = e^{-a_2 \Delta \eta / 2} \quad \text{and} \quad t_2 = e^{b_2 \Delta \eta / 2} \quad (\text{D8})$$

Three expressions for F_{j-1} , F_m , and F_{nm} can also be obtained using the expression for F in the lower interval. These expressions will be similar to the R.H.S. of (D3), (D5), and (D7) with the subscript "2" replaced by "1" to refer to point $j-1$. On the other hand the expression for F in the upper interval will give for F_{j+1} , F_n , and F_{nn} expressions that are similar to the R.H.S. of (D3), (D4), and (D6) if the subscript "2" is replaced by "3" to refer to the point $j+1$.

The resulting set of 11 equations is sufficient to eliminate A_1 , B_1 , A_2 , B_2 , A_3 , B_3 , F_m , F_n , F_{nm} , and F_{nn} and an equation that involves F_{j-1} , F_j , and F_{j+1} , only is obtained

$$\begin{aligned} (g_1 - h_1)e_3 F_{j+1} - (g_1 h_3 - g_3 h_1)F_j + (h_3 - g_3)e_1 F_{j-1} \\ = (g_1 - h_1)d_3 - (g_1 h_3 - g_3 h_1)f_2 + (h_3 - g_3)d_1 \end{aligned} \quad (\text{D9})$$

where

$$d_1 = (a_1 - b_1 r_1 t_1) t_2 f_2 + [(a_1 - b_1) t_1 - (a_1 - b_1 r_1 t_1)] t_2 f_1$$

$$d_3 = (a_3 r_3 t_3 - b_3) r_2 f_2 + [(a_3 - b_3) r_3 - (a_3 r_3 t_3 - b_3)] r_2 f_3$$

$$e_1 = t_1 t_2 (a_1 - b_1)$$

$$e_3 = r_2 r_3 (a_3 - b_3)$$

ORIGINAL PAGE IS
OF POOR QUALITY

$$g_1 = [(a_1 - b_1 r_1 t_1) - a_2 (1 - r_1 t_1)] r_2 t_2$$

$$g_3 = [(a_3 r_3 t_3 - b_3) + a_2 (1 - r_3 t_3)]$$

$$h_1 = [(a_1 - b_1 r_1 t_1) - b_2 (1 - r_1 t_1)]$$

$$h_3 = [(a_3 r_3 t_3 - b_3) + b_2 (1 - r_3 t_3)] r_2 t_2 \quad (D10)$$

The scheme given by equation (D9) has the advantage over the scheme given by equation (58) that no values need be evaluated at the midpoints m and n .

Instead of using the Thomas algorithm to invert the set of three point difference equations given in (D9) a simpler inversion scheme can be designed to first evaluate the coefficients A_j and B_j ($j = 1, \dots, N + 1$) and then the values of F and its derivatives can be calculated directly from (D1) in the appropriate interval. Such an algorithm eliminates the tedious algebraic steps necessary to derive the tridiagonal equation (D9) and is described here for future reference.

The requirement that F_m and F_{nm} be continuous at point m , the midpoint between the $j-1$ and j th points, provides two equations that involve A_{j-1} , B_{j-1} , A_j , and B_j that can be solved for A_j and B_j . Thus one can write

$$A_j = \alpha_{1j} A_{j-1} + \beta_{1j} B_{j-1} + \gamma_{1j} \quad (D11)$$

$$B_j = \alpha_{2j} A_{j-1} + \beta_{2j} B_{j-1} + \gamma_{2j} \quad (D12)$$

where the α 's, β 's, and γ 's are all known.

Now assuming that A_j and B_j ($j=1, \dots, N+1$) can be written in the form

$$A_j = \mu_{1j} A_1 + v_{1j} \quad (D13)$$

$$B_j = \mu_{2j} A_1 + v_{2j} \quad (D14)$$

and substituting A_{j-1} and B_{j-1} into (D11) and (D12) one gets

$$A_j = (\alpha_{1j} \mu_{1,j-1} + \beta_{1j} \mu_{2,j-1}) A_1 + (\alpha_{1j} v_{1,j-1} + \beta_{1j} v_{2,j-1} + \gamma_{1j}) \quad (D15)$$

$$B_j = (\alpha_{2j} \mu_{1,j-1} + \beta_{2j} \mu_{2,j-1}) A_1 + (\alpha_{2j} v_{1,j-1} + \beta_{2j} v_{2,j-1} + \gamma_{2j}) \quad (D16)$$

Comparing (D15) and (D16) with (D13) and (D14) the following recurrence relations for μ_1 , v_1 , μ_2 , and v_2 are obtained

$$\begin{aligned} \mu_{1j} &= \alpha_{1j} \mu_{1,j-1} + \beta_{1j} \mu_{2,j-1} \\ v_{1j} &= \alpha_{1j} v_{1,j-1} + \beta_{1j} v_{2,j-1} + \gamma_{1j} \\ \mu_{2j} &= \alpha_{2j} \mu_{1,j-1} + \beta_{2j} \mu_{2,j-1} \\ v_{2j} &= \alpha_{2j} v_{1,j-1} + \beta_{2j} v_{2,j-1} + \gamma_{2j} \end{aligned} \quad (D17)$$

At $j = 1$ the following relations are known

$$\mu_{1,1} = 1 \quad (D18)$$

$$v_{1,1} = 0 \quad (D19)$$

$$F_1 = A_1 + B_1 + f_1 \quad (D20)$$

ORIGINAL PAGE IS
OF POOR QUALITY

while at $j = N+1$ the following relations are valid

$$A_{N+1} = \mu_{1,N+1} A_1 + v_{1,N+1} \quad (D21)$$

$$B_{N+1} = \mu_{2,N+1} A_1 + v_{2,N+1} \quad (D22)$$

$$F_{N+1} = A_{N+1} + B_{N+1} + f_{N+1} \quad (D23)$$

In (D20) and (D23), F_1 and F_{N+1} are assumed to be known as boundary conditions of the differential equation under consideration. Other types of boundary conditions can be treated using expression (D1) for F to give two relations involving (A_1, B_1) and (A_{N+1}, B_{N+1}) to replace (D20) and (D23) respectively.

The calculation procedure will be as follows:

1. Use (D20) to obtain $\mu_{2,1}$ and $v_{2,1}$ by comparing it with (D14) for $j=1$.
2. Use the recurrence relations (D17) to calculate μ_{1j} , v_{1j} , μ_{2j} , and v_{2j} for $j=2, \dots, N+1$ knowing their values for $j=1$.
3. Knowing the values of $\mu_{1,N+1}$, $v_{1,N+1}$, $\mu_{2,N+1}$, and $v_{2,N+1}$ solve equations (D21), (D22) and (D23) to determine A_1 .
4. Calculate the A_j 's and B_j 's, $j=2, \dots, N+1$ using (D13) and (D14).
5. Calculate F and its derivatives using (D1).

η	C	CC	UW
0.0000	0.00000000000000	0.00000000000000	0.00000000000000
0.5000	0.2357019175320	0.2330208993764	0.2351076811025
1.0000	0.4645601396245	0.4593522776107	0.4646173272125
1.5000	0.6681224180019	0.6609601688855	0.6682772996516
2.0000	0.8250732563570	0.8170778673049	0.8232959968979
2.5000	0.9248680267167	0.9175782221355	0.9205713905362
3.0000	0.9749164338430	0.9696712237985	0.9700708297523
3.5000	0.9938657448810	0.9910125484759	0.9905255765472
4.0000	0.9989977732033	0.9978690991754	0.9974630685493
4.5000	0.9999093638850	0.9995958195990	0.9994204753158
5.0000	0.9999977293927	0.9999382939866	0.9998800609093
5.5000	1.0000000763307	0.9999923163958	0.9999805554494
6.0000	0.999999934052	0.999992026792	0.9999970966843
6.5000	1.0000000008841	0.999999289259	0.999996179303
7.0000	0.999999998435	0.999999943462	0.9999999553871
7.5000	1.0000000000340	0.999999995812	0.9999999953493
8.0000	0.999999999915	0.999999999699	0.9999999995647
8.5000	1.0000000000025	0.999999999978	0.999999999632
9.0000	0.999999999993	0.999999999998	0.999999999972
9.5000	1.0000000000004	1.0000000000000	0.999999999998
10.0000	1.0000000000000	1.0000000000000	1.0000000000000

TABLE 1: BLASIUS FLOW
VELOCITY PROFILES FOR FINITE DIFFERENCE SCHEMES

η	RUDR & TPE	EB
0.0	0.0	0.0
0.5000	0.2330545736248	0.2354540274489
1.0000	0.4594177063782	0.4624499328736
1.5000	0.6610505304537	0.6632486160505
2.0000	0.8171816649639	0.8179955191227
2.5000	0.9176822625149	0.9175423945048
3.0000	0.9697617889022	0.9693773037453
3.5000	0.9910775955934	0.9908197225985
4.0000	0.9979055192255	0.9977999877228
4.5000	0.9996112085620	0.9995811491306
5.0000	0.9999431583436	0.9999369107351
5.5000	0.9999934731040	0.9999925041092
6.0000	0.9999994125816	0.9999992990353
6.5000	0.9999999586259	0.9999999484974
7.0000	0.999999977220	0.999999970308
7.5000	0.999999999020	0.999999998658
8.0000	0.999999999967	0.999999999953
8.5000	0.999999999999	0.999999999999
9.0000	1.0000000000000	1.0000000000000
9.5000	1.0000000000000	1.0000000000000
10.0000	1.0000000000000	1.0000000000000

TABLE 2: BLASIUS FLOW
VELOCITY PROFILES FOR THE EXPONENTIAL SCHEMES

η	C	CC	UW
0.0000	0.00000000000000	0.00000000000000	0.00000000000000
1.0000	0.1250001385162	0.1249999975309	0.1249999909974
2.0000	0.2499997050946	0.2499999645076	0.2499998742456
3.0000	0.3750012121533	0.3749994929750	0.3749984090375
4.0000	0.4999949574309	0.4999923489294	0.4999814704162
5.0000	0.6250309063076	0.6248863622985	0.6248108847319
6.0000	0.7495845282511	0.7486180257022	0.7484101762581
7.0000	0.8655708850288	0.8640494195098	0.8647799101187
8.0000	0.9511730093339	0.9501065605221	0.9539534103003
9.0000	0.9912926329848	0.9898745947980	0.9907807947510
10.0000	0.9999098573588	0.9989944930587	0.9988564309871
11.0000	1.0000116855790	0.9999404691035	0.9999059459089
12.0000	0.9999971727840	0.999963154787	0.999945558809
13.0000	1.0000009469873	0.9999996758827	0.999997674294
14.0000	0.999996238077	0.999999594763	0.99999923875
15.0000	1.0000001848676	0.999999932714	0.999999998033
16.0000	0.999999164921	0.999999986084	0.999999999959
17.0000	1.0000000599589	0.999999996617	0.999999999999
18.0000	0.999999799747	0.999999999103	1.000000000000
19.0000	1.0000000297870	0.999999999785	1.000000000000
20.0000	1.0000000000000	1.0000000000000	1.0000000000000

TABLE 3
MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT
VELOCITY PROFILES FOR FINITE DIFFERENCE SCHEMES

η	RUDR	TPE	EB
0.0	0.0	0.0	0.0
1.0000	0.1249999999508	0.1255323119837	0.1251331647276
2.0000	0.2499999973900	0.2515652915513	0.2502705990423
3.0000	0.3749998831525	0.3780471038645	0.3754152395157
4.0000	0.4999960124792	0.5048612680820	0.5005658740969
5.0000	0.6249072666931	0.6316647394431	0.6256233708841
6.0000	0.7486703126367	0.7567270901976	0.7493524451070
7.0000	0.8640974059548	0.8715107771055	0.8641102486477
8.0000	0.9501418039624	0.9544158150455	0.9492358621257
9.0000	0.9899214551605	0.9911497573511	0.9892008319580
10.0000	0.9990531540856	0.9992034627362	0.988872841214
11.0000	0.9999627371932	0.9999699736813	0.9999502645726
12.0000	0.9999994186416	0.9999995509957	0.999991012758
13.0000	0.999999965059	0.999999974114	0.999999937433
14.0000	0.999999999920	0.999999999943	0.999999999838
15.0000	1.0000000000000	1.0000000000000	1.0000000000000
16.0000	1.0000000000000	1.0000000000000	1.0000000000000
17.0000	1.0000000000000	1.0000000000000	1.0000000000000
18.0000	1.0000000000000	1.0000000000000	1.0000000000000
19.0000	1.0000000000000	1.0000000000000	1.0000000000000
20.0000	1.0000000000000	1.0000000000000	1.0000000000000

TABLE 4
MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT
VELOCITY PROFILES FOR THE EXPONENTIAL SCHEMES

n	C	CC	UW
0.0000	0.00000000000000	0.00000000000000	0.00000000000000
2.0000	0.0399946191128	0.0399999985266	0.04000000000000
4.0000	0.0800004671197	0.0799999966522	0.08000000000000
6.0000	0.1199940901971	0.1199999942626	0.12000000000000
8.0000	0.1600010729529	0.1599999912057	0.16000000000000
10.0000	0.1999933981039	0.1999999872766	0.20000000000000
12.0000	0.2400018778280	0.2399999821961	0.24000000000000
14.0000	0.2799924667368	0.2799999755781	0.28000000000000
16.0000	0.3200029792111	0.3199999668805	0.32000000000000
18.0000	0.3599911701180	0.3599999553297	0.36000000000000
20.0000	0.4000045428948	0.3999999397991	0.40000000000000
22.0000	0.4399892873270	0.4399999186133	0.44000000000000
24.0000	0.4800068696053	0.4799998892170	0.48000000000000
26.0000	0.5199864018206	0.5199998476014	0.52000000000000
28.0000	0.5600105515772	0.5599997872657	0.56000000000000
30.0000	0.5999816505634	0.5999996972513	0.60000000000000
32.0000	0.6400168896633	0.6399995582156	0.64000000000000
34.0000	0.6799730012435	0.6799993340641	0.68000000000000
36.0000	0.7200292115698	0.7199989526665	0.72000000000000
38.0000	0.7599546971966	0.7599982568631	0.76000000000000
40.0000	0.8000582225960	0.7999968634183	0.7999999999994
42.0000	0.8399048812150	0.8399936833930	0.8399999998945
44.0000	0.8801548127682	0.8799848601666	0.8799999826392
46.0000	0.9196792203024	0.9199510719014	0.9199980557216
48.0000	0.9608975687037	0.9597100879897	0.9598707621555
50.0000	0.9935820005269	0.9931702198606	0.9958921809931
52.0000	1.0008980904585	0.9997149261997	0.9998778987386
54.0000	0.9996811278535	0.9999503497977	0.9999983374438
56.0000	1.0001560202453	0.9999840005096	0.999999871412
58.0000	0.9999066937334	0.9999930172200	0.99999999999361
60.0000	1.0000595721305	0.9999963613360	0.9999999999998
62.0000	0.9999563522191	0.9999978724550	1.00000000000000
64.0000	1.0000305505276	0.9999986521368	1.00000000000000
66.0000	0.9999744813318	0.9999990946317	1.00000000000000
68.0000	1.0000181516110	0.9999993644662	1.00000000000000
70.0000	0.9999829476229	0.9999995385277	1.00000000000000
72.0000	1.0000117006454	0.9999996560146	1.00000000000000
74.0000	0.9999875111227	0.9999997383396	1.00000000000000
76.0000	1.0000078837714	0.9999997978756	1.00000000000000
78.0000	0.9999902057064	0.9999998421129	1.00000000000000
80.0000	1.0000054074186	0.9999998757682	1.00000000000000
82.0000	0.9999918952477	0.9999999019127	1.00000000000000
84.0000	1.0000036835870	0.9999999226055	1.00000000000000
86.0000	0.9999929967801	0.9999999392623	1.00000000000000
88.0000	1.0000024141944	0.9999999528787	1.00000000000000
90.0000	0.9999937315261	0.9999999641689	1.00000000000000
92.0000	1.0000014351876	0.9999999736543	1.00000000000000
94.0000	0.9999942256535	0.9999999817216	1.00000000000000
96.0000	1.0000006502884	0.9999999886621	1.00000000000000
98.0000	0.9999945553756	0.9999999946978	1.00000000000000
100.0000	1.00000000000000	1.00000000000000	1.00000000000000

TABLE 5
STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT
VELOCITY PROFILES FOR FINITE DIFFERENCE SCHEMES

η	RUDR	TPE	ER
0.0	0.0	0.0	0.0
2.0000	0.04000000000000	0.0400307560792	0.0400006167749
4.0000	0.08000000000000	0.0800923608737	0.0800012187300
6.0000	0.12000000000000	0.1201849905028	0.1200017875510
8.0000	0.16000000000000	0.1603089127082	0.1600023039530
10.0000	0.20000000000000	0.2004644872605	0.2000027470678
12.0000	0.24000000000000	0.2406521713189	0.2400030937425
14.0000	0.28000000000000	0.2808725254664	0.2800033177037
16.0000	0.32000000000000	0.3211262212638	0.3200033885324
18.0000	0.36000000000000	0.3614140500788	0.3600032703770
20.0000	0.40000000000000	0.4017369331309	0.4000029203098
22.0000	0.44000000000000	0.4420959321954	0.4400022862057
24.0000	0.48000000000000	0.4824922597755	0.4800013039912
26.0000	0.52000000000000	0.5229272861505	0.5199998940945
28.0000	0.56000000000000	0.5634025378059	0.5599919569605
30.0000	0.60000000000000	0.6039196755681	0.5999953077714
32.0000	0.64000000000000	0.6444804270795	0.6399919711466
34.0000	0.68000000000000	0.6850864163865	0.6799875794772
36.0000	0.72000000000000	0.7257387541931	0.7199819860905
38.0000	0.76000000000000	0.7664370380782	0.7599750306494
40.0000	0.80000000000000	0.8071767653865	0.7999668431905
42.0000	0.84000000000000	0.8479419049149	0.8399587288480
44.0000	0.87999999999999	0.8886797806307	0.8799504814003
46.0000	0.9199999785978	0.9291924545652	0.9199820069041
48.0000	0.9599317532875	0.9683961219803	0.9599439967085
50.0000	0.9932522113264	0.9960251114147	0.9923219715474
52.0000	0.9999367431587	0.9999742948492	0.9998746822664
54.0000	0.9999999854915	0.9999999959049	0.9999999902784
56.0000	0.99999999999999	1.00000000000000	0.99999999999999
58.0000	1.00000000000000	1.00000000000000	1.00000000000000
60.0000	1.00000000000000	1.00000000000000	1.00000000000000
62.0000	1.00000000000000	1.00000000000000	1.00000000000000
64.0000	1.00000000000000	1.00000000000000	1.00000000000000
66.0000	1.00000000000000	1.00000000000000	1.00000000000000
68.0000	1.00000000000000	1.00000000000000	1.00000000000000
70.0000	1.00000000000000	1.00000000000000	1.00000000000000
72.0000	1.00000000000000	1.00000000000000	1.00000000000000
74.0000	1.00000000000000	1.00000000000000	1.00000000000000
76.0000	1.00000000000000	1.00000000000000	1.00000000000000
78.0000	1.00000000000000	1.00000000000000	1.00000000000000
80.0000	1.00000000000000	1.00000000000000	1.00000000000000
82.0000	1.00000000000000	1.00000000000000	1.00000000000000
84.0000	1.00000000000000	1.00000000000000	1.00000000000000
86.0000	1.00000000000000	1.00000000000000	1.00000000000000
88.0000	1.00000000000000	1.00000000000000	1.00000000000000
90.0000	1.00000000000000	1.00000000000000	1.00000000000000
92.0000	1.00000000000000	1.00000000000000	1.00000000000000
94.0000	1.00000000000000	1.00000000000000	1.00000000000000
96.0000	1.00000000000000	1.00000000000000	1.00000000000000
98.0000	1.00000000000000	1.00000000000000	1.00000000000000
100.0000	1.00000000000000	1.00000000000000	1.00000000000000

TABLE 6
STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT
VELOCITY PROFILES FOR THE EXPONENTIAL SCHEMES

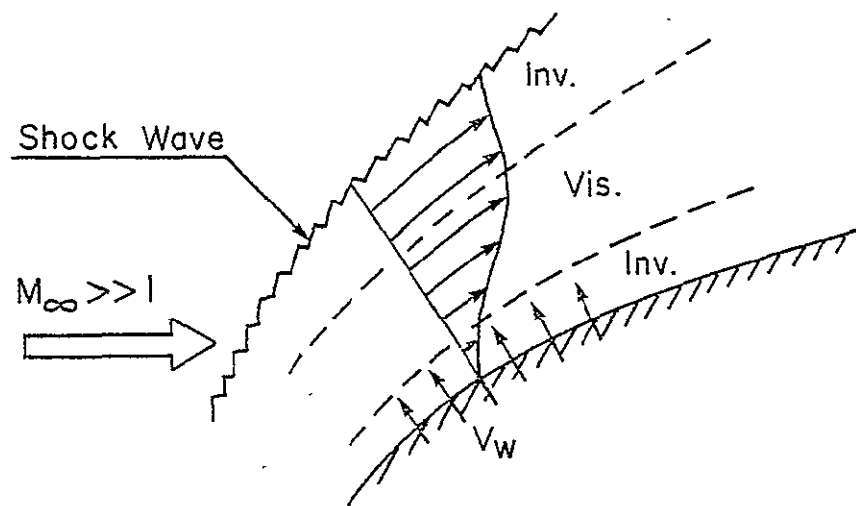


FIGURE 1. VISCID/INVISCID FLUID FLOW
OVER A REENTRY BODY

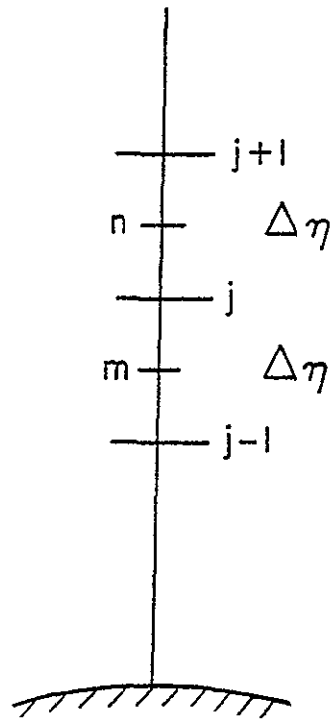


FIGURE 2: THREE-POINT UNIFORM GRID CONFIGURATION

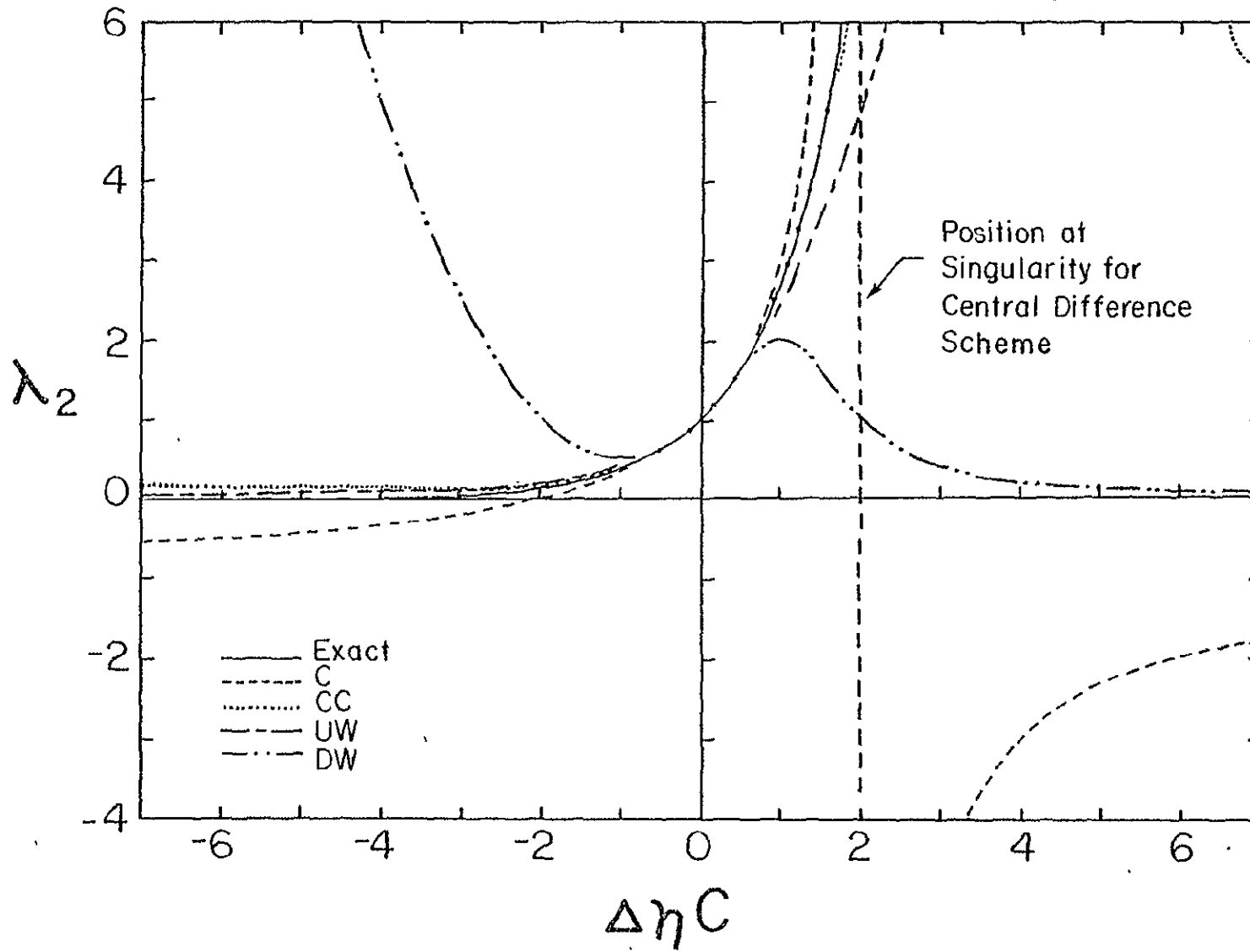


FIGURE 3: CHARACTERISTIC ROOT λ_2 OF FINITE DIFFERENCE EQUATIONS

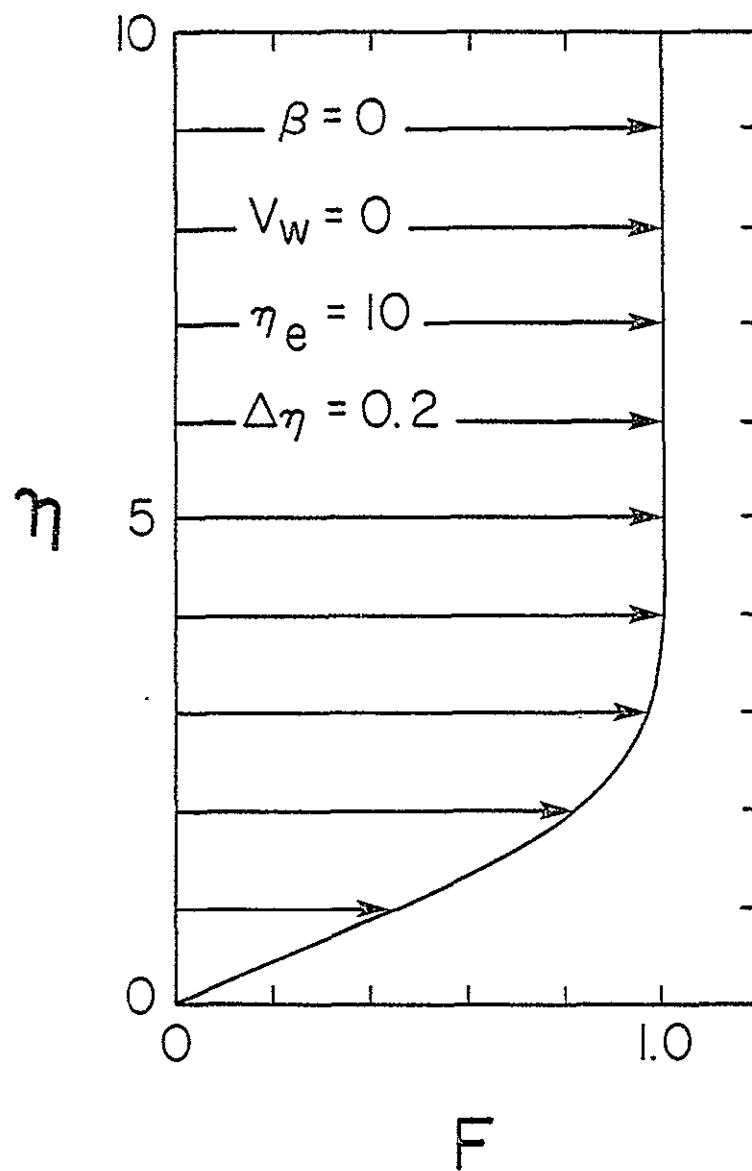


FIGURE 4: BLASIUS FLOW-VELOCITY PROFILE

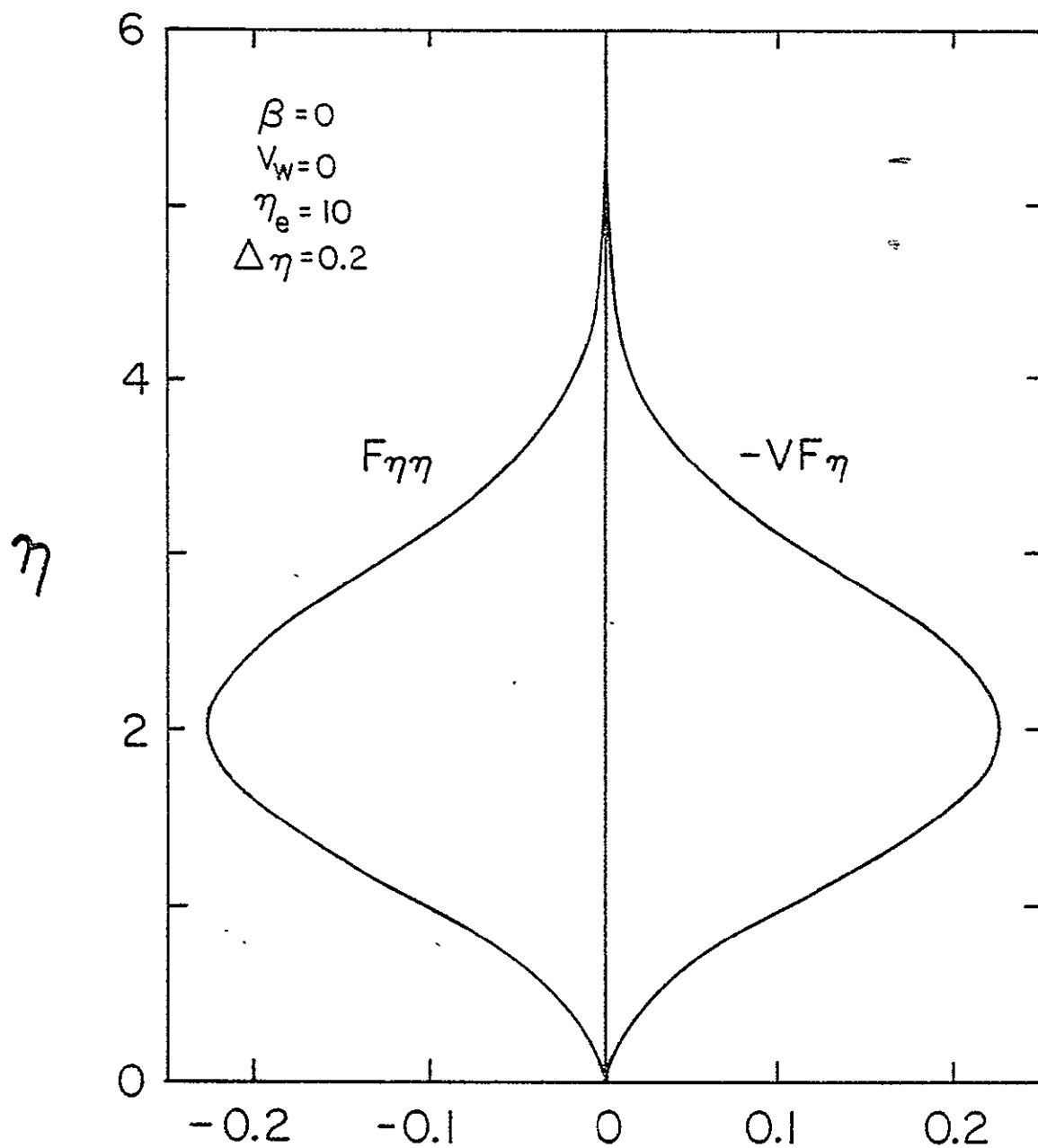


FIGURE 5: BLASIUS FLOW-MOMENTUM BALANCE

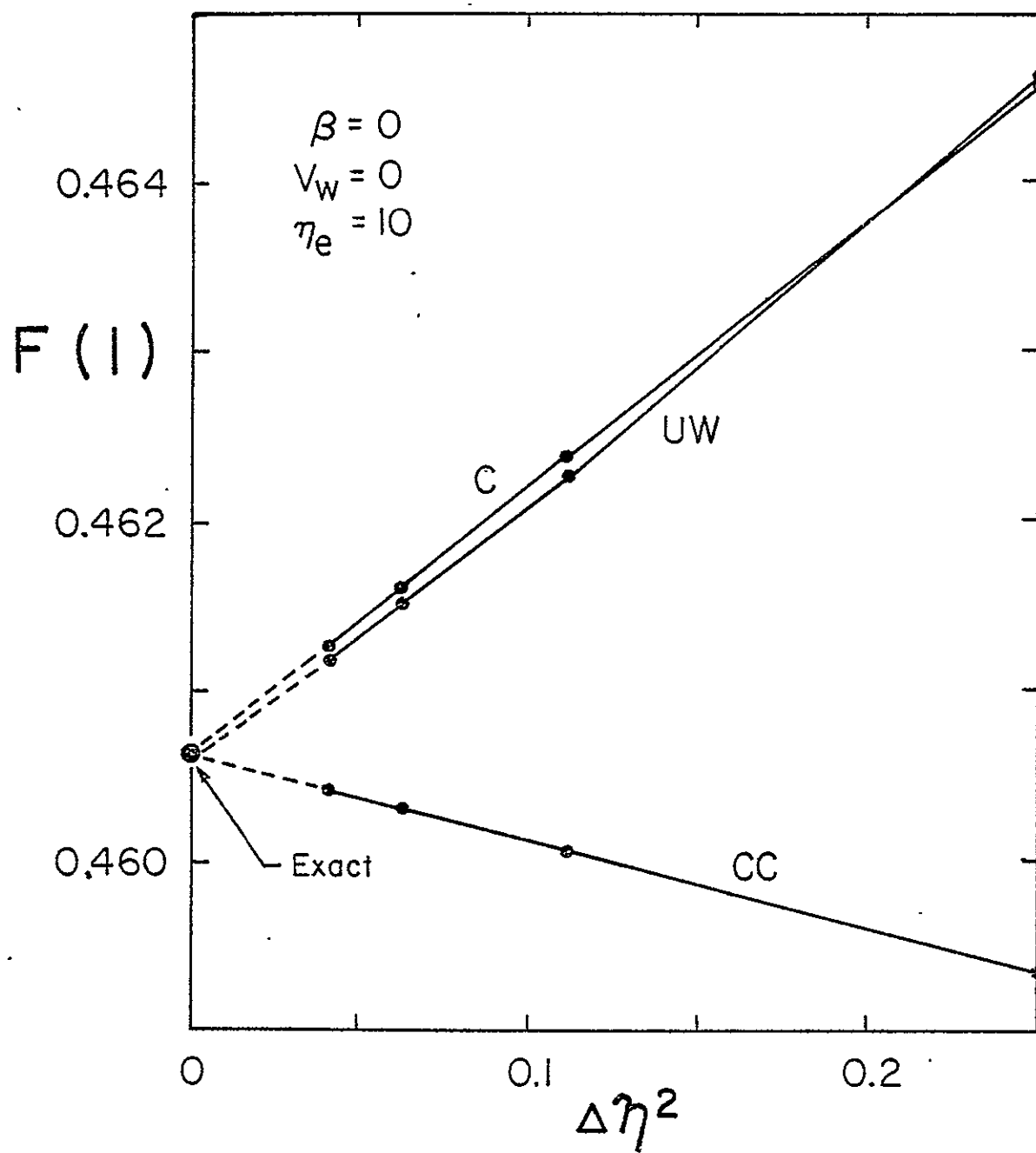


FIGURE 6: BLASIOUS FLOW-STEP SIZE STUDY FOR FINITE DIFFERENCE SCHEMES, F AT $\eta = 1.0$

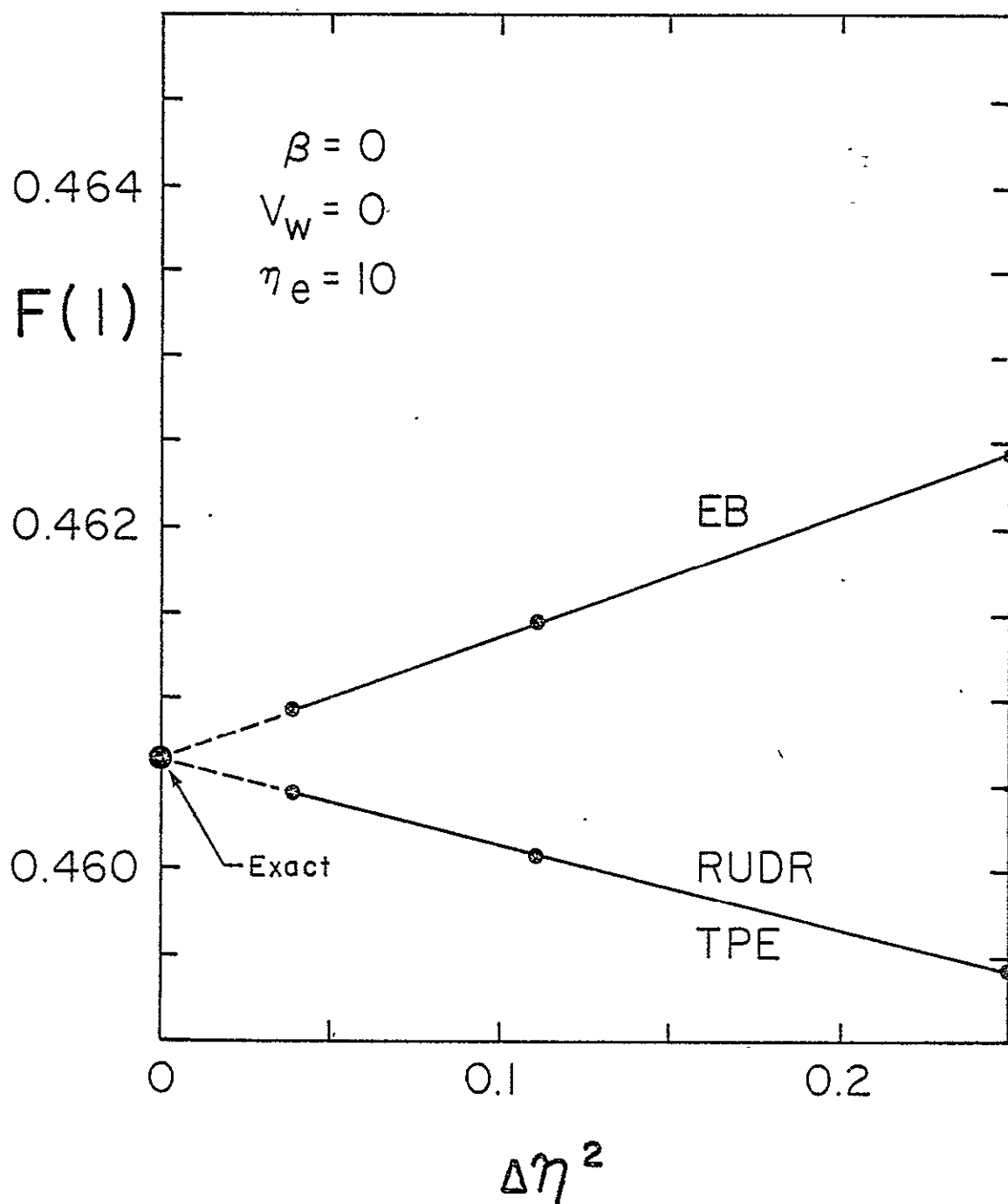


FIGURE 7: BLASIUS FLOW-STEP SIZE STUDY FOR THE EXPONENTIAL SCHEMES, F AT $\eta = 1.0$

ORIGINAL PAGE IS
OF POOR QUALITY

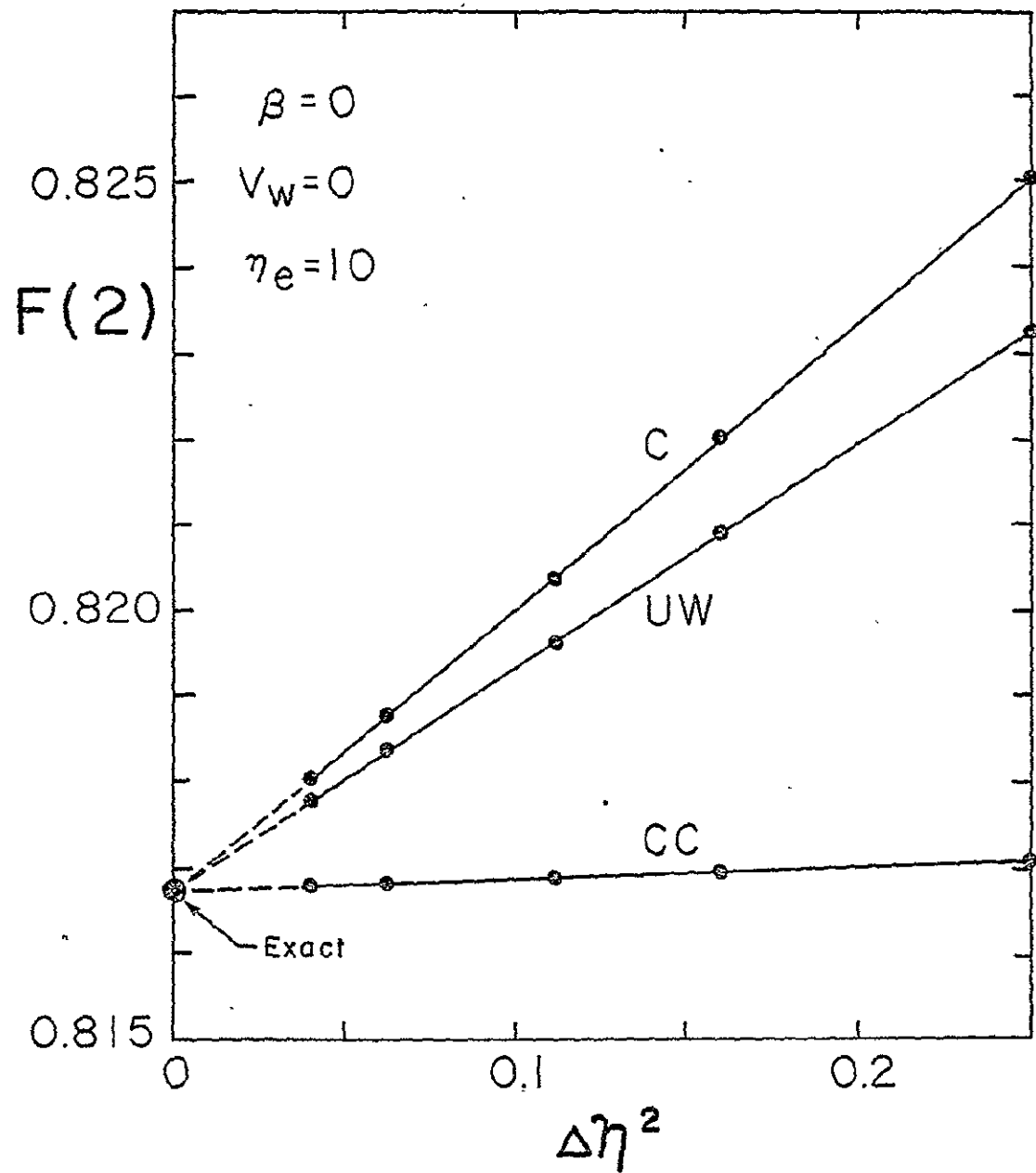


FIGURE 8: BLASIUS FLOW-STEP SIZE STUDY FOR FINITE DIFFERENCE SCHEMES, F AT $\eta = 2.0$

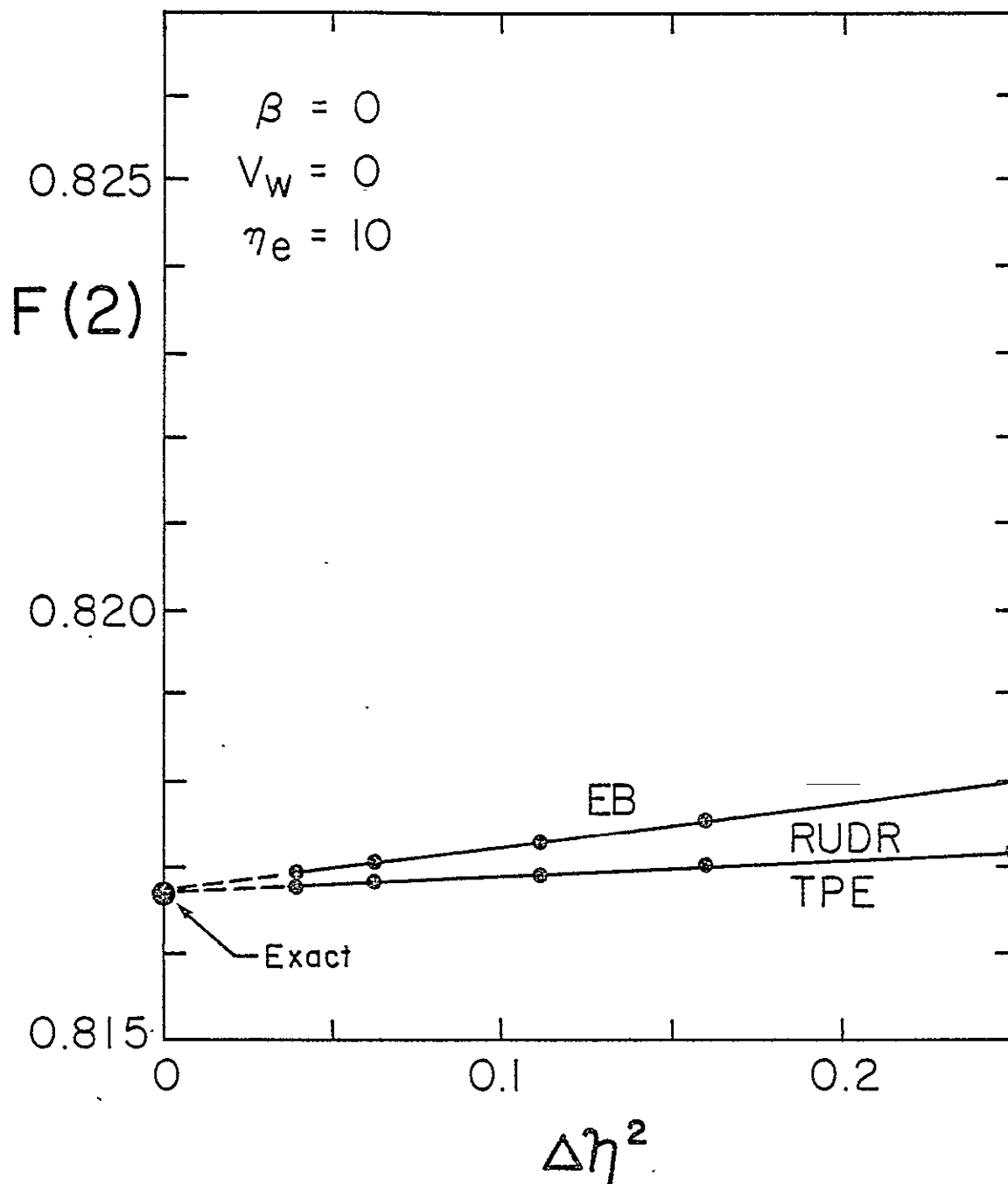


FIGURE 9: BLASIUS FLOW-STEP SIZE STUDY FOR THE EXPONENTIAL SCHEMES, F AT $\eta = 2.0$

ORIGINAL PAGE IS
OF POOR QUALITY

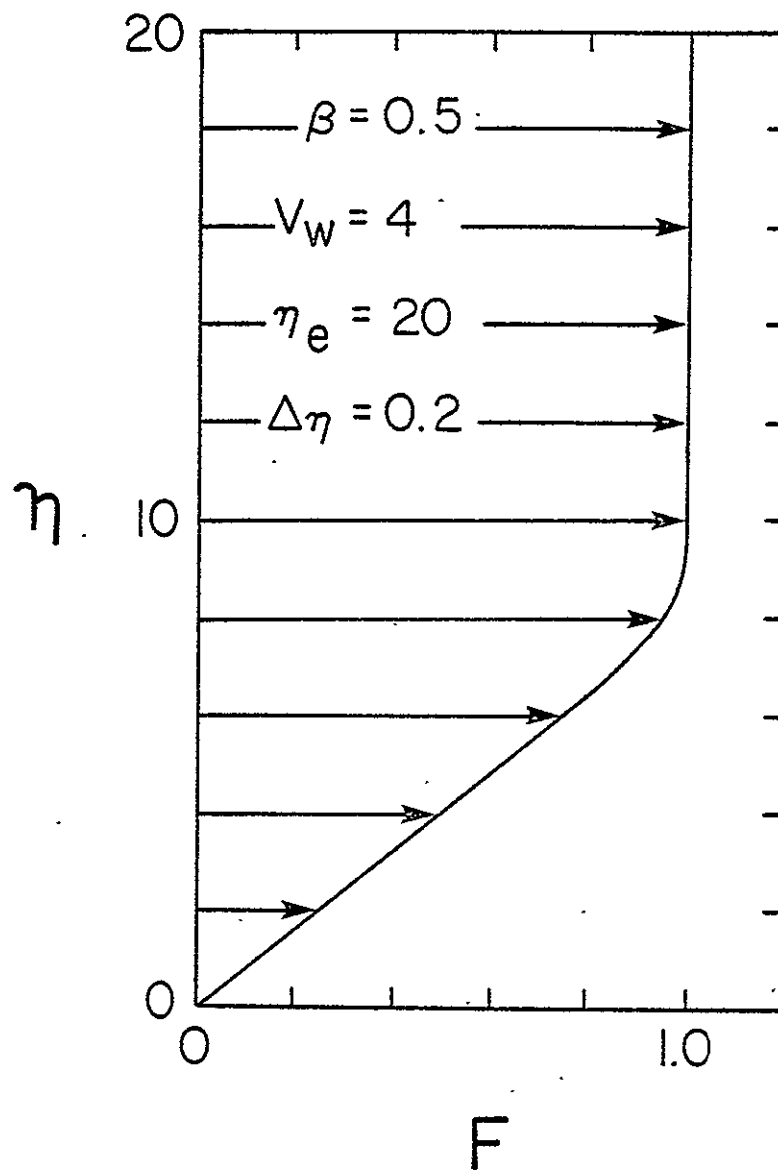


FIGURE 10: MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT-VELOCITY PROFILE

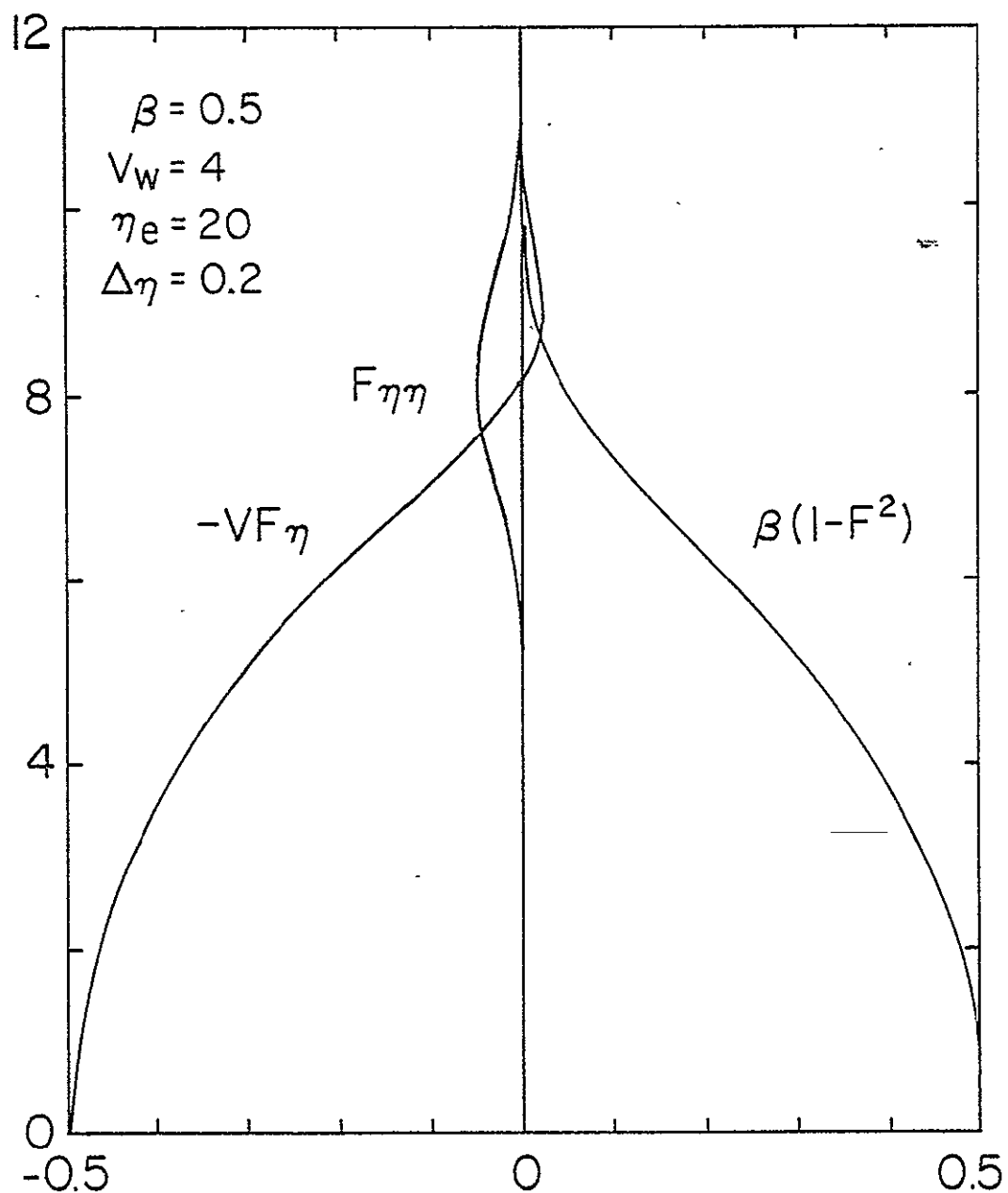


FIGURE 11: MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT-MOMENTUM BALANCE

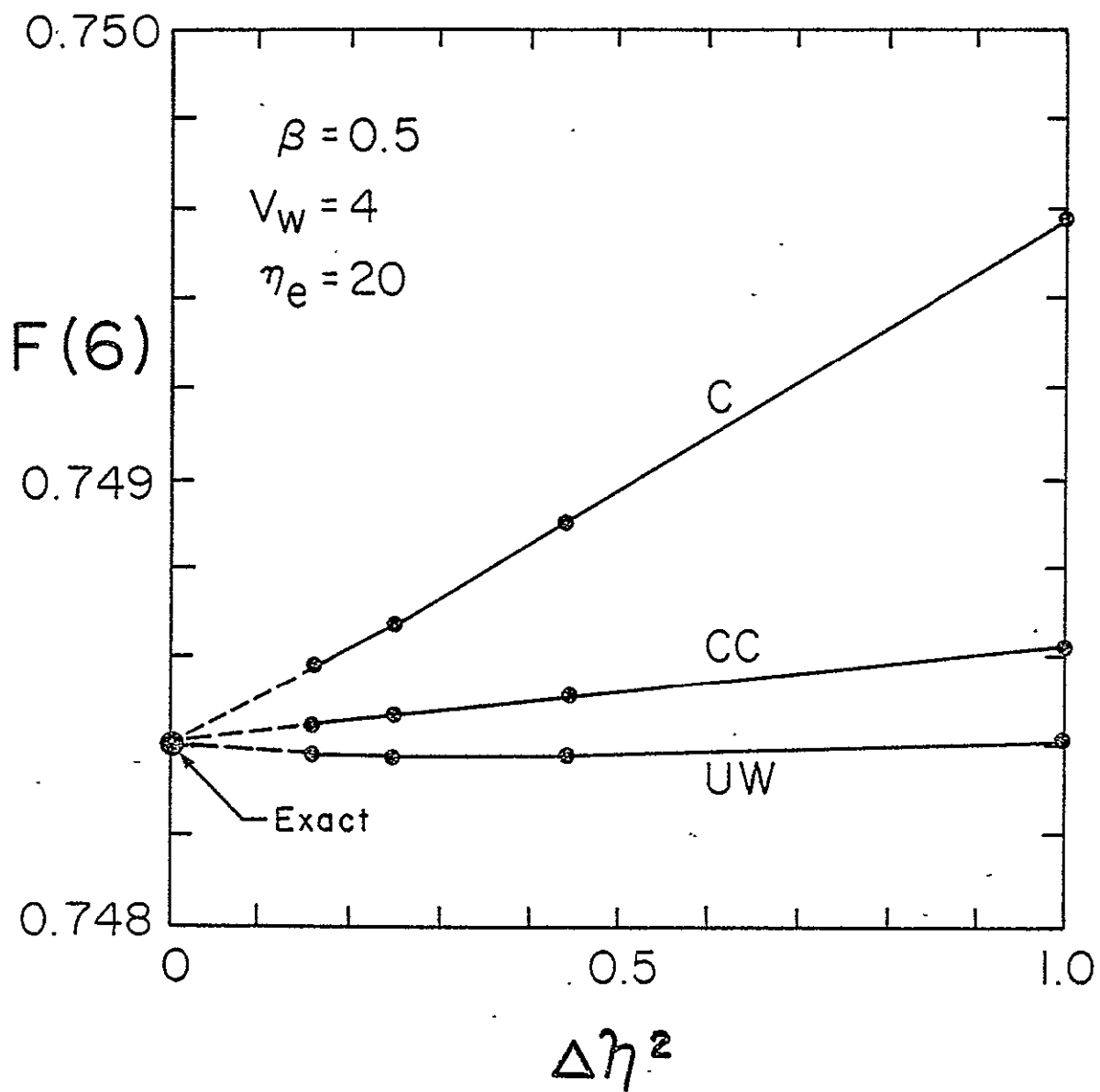


FIGURE 12: MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT—STEP SIZE STUDY FOR FINITE DIFFERENCE SCHEMES, F AT $\eta = 6.0$

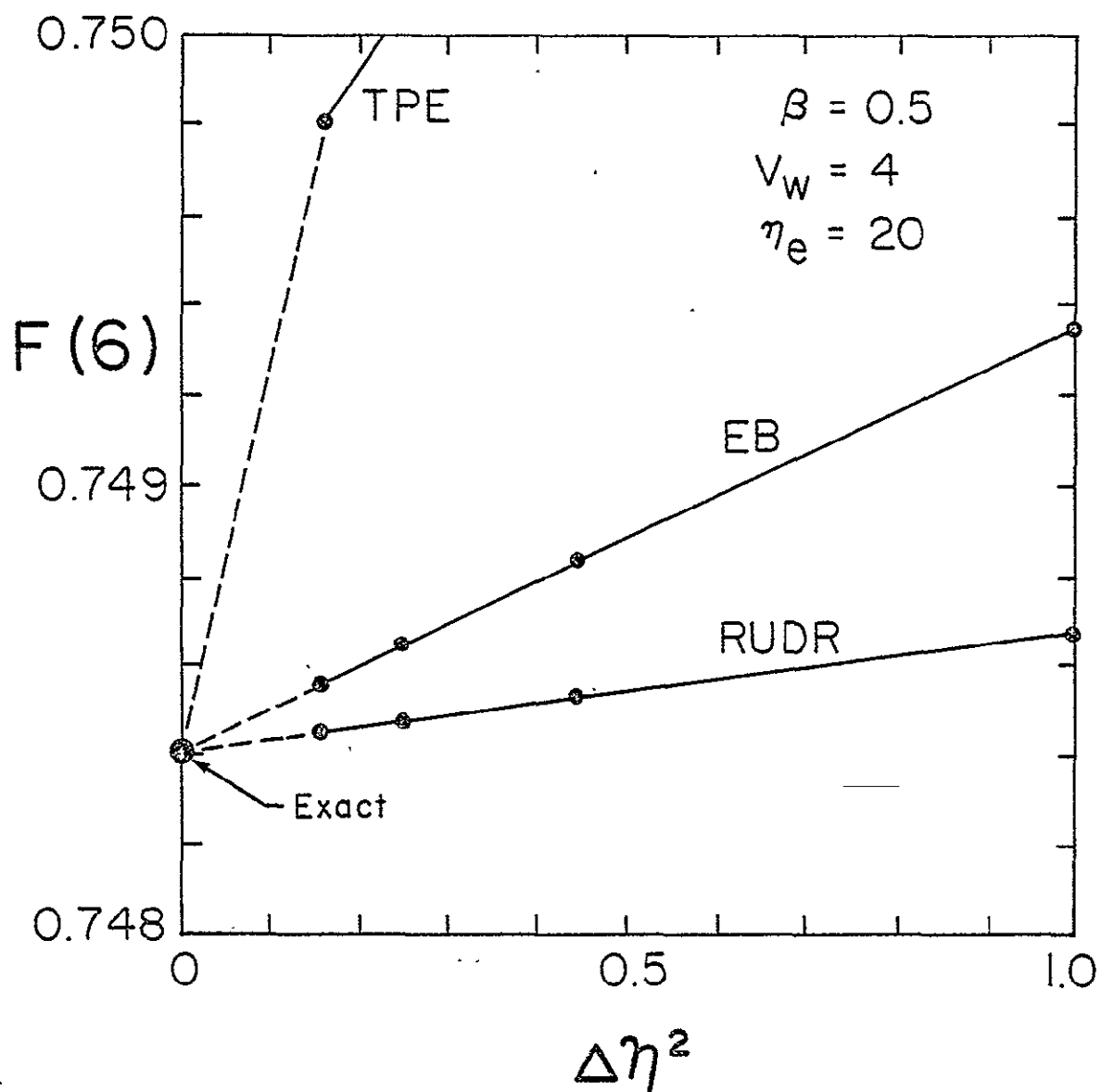


FIGURE 13: MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT-STEP SIZE STUDY FOR THE EXPONENTIAL SCHEME, F AT $\eta = 6.0$

ORIGINAL PAGE IS
OF POOR QUALITY

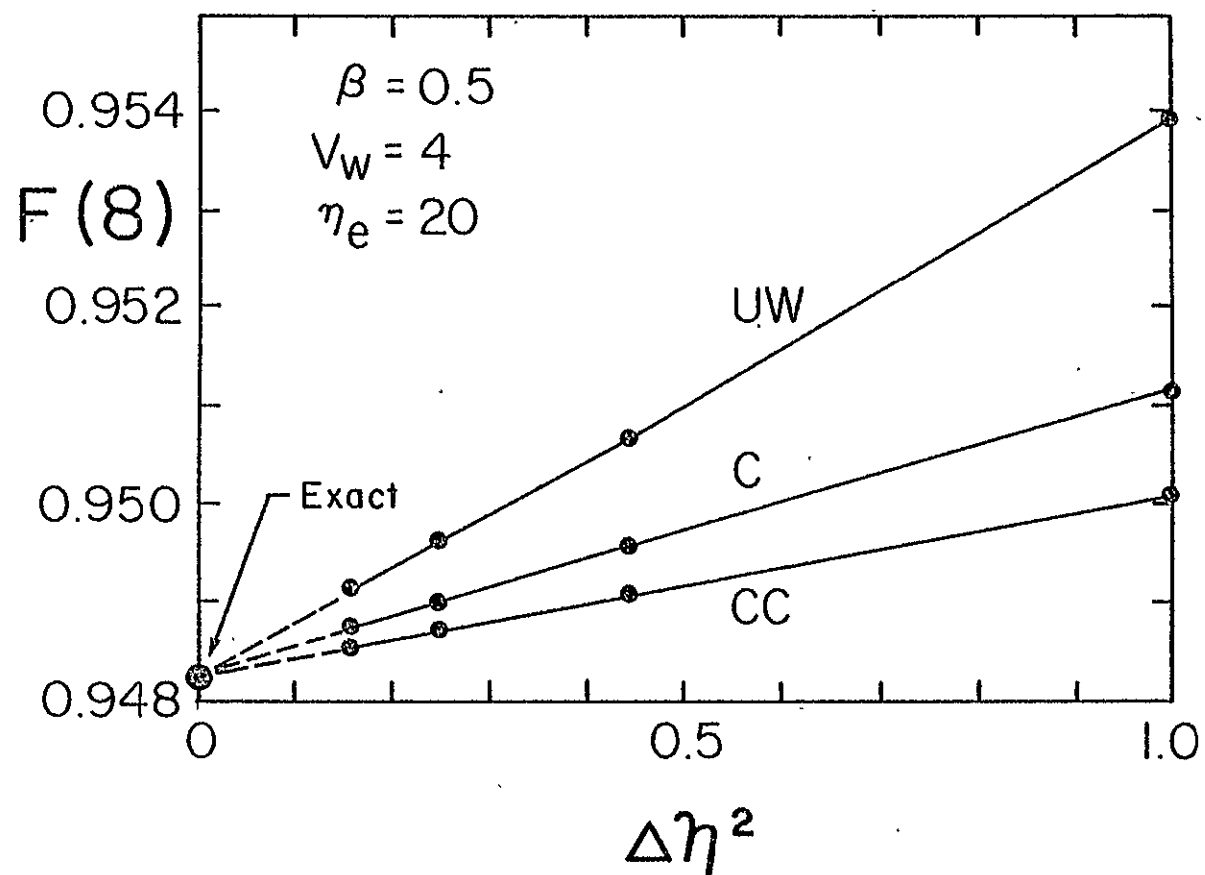


FIGURE 14: MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT-STEP
 SIZE STUDY FOR FINITE DIFFERENCE SCHEMES, F AT $\eta = 8.0$

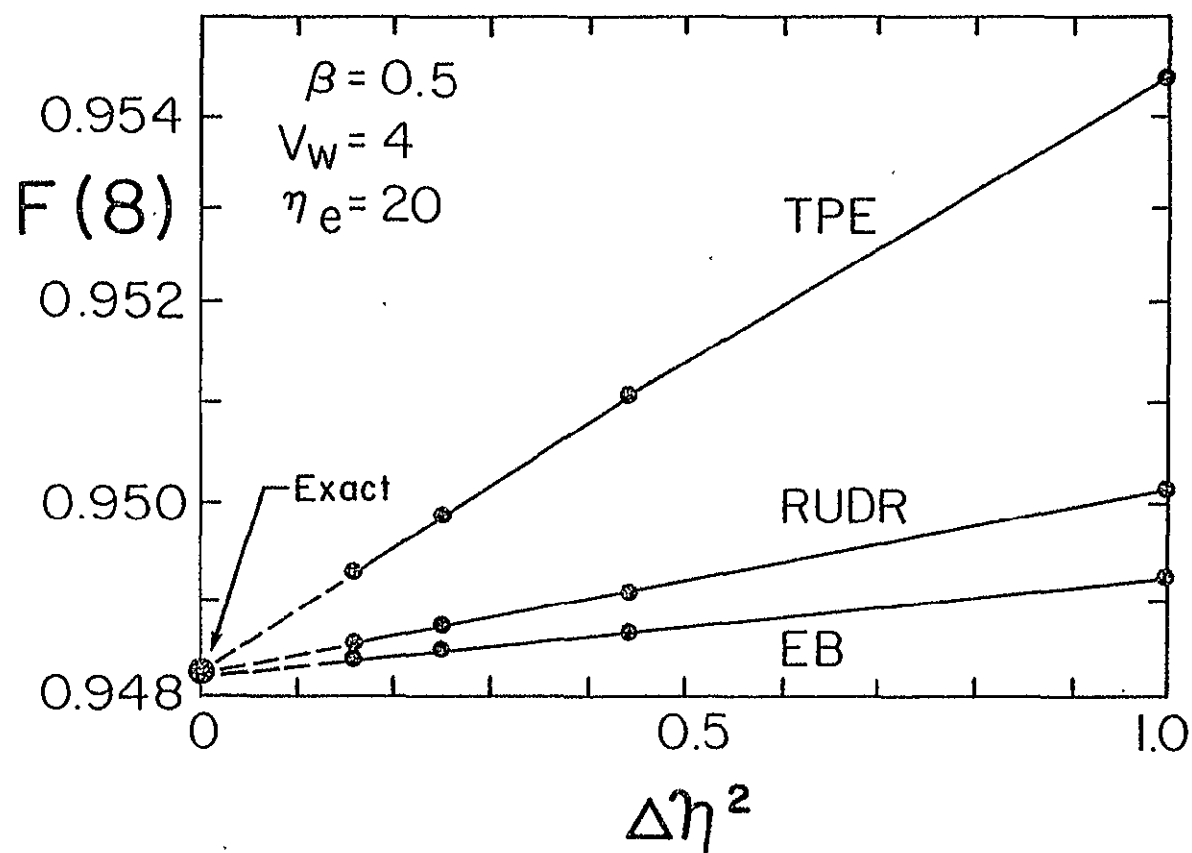


FIGURE 15: MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT-STEP SIZE STUDY FOR THE EXPONENTIAL SCHEMES, F AT $\eta = 8.0$

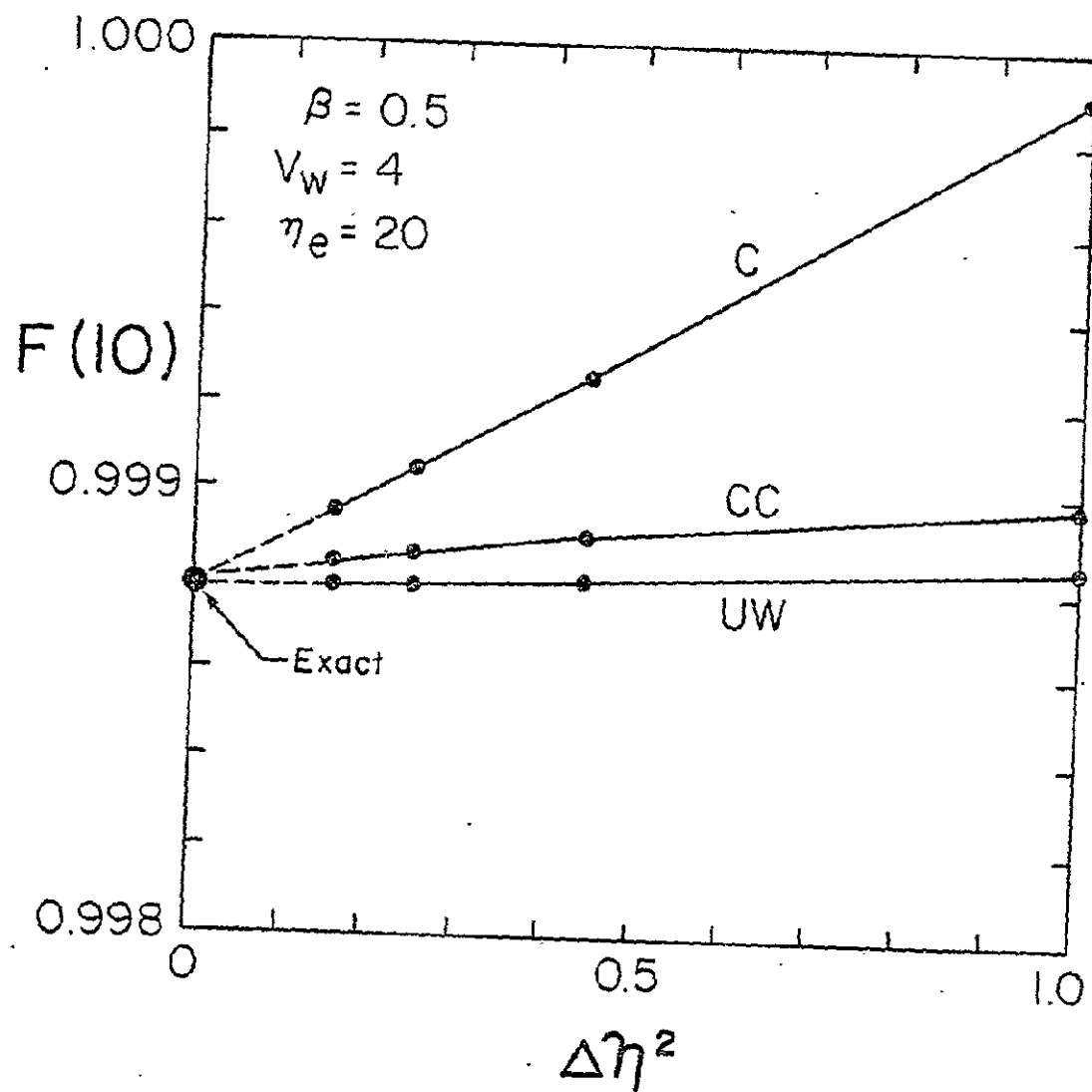
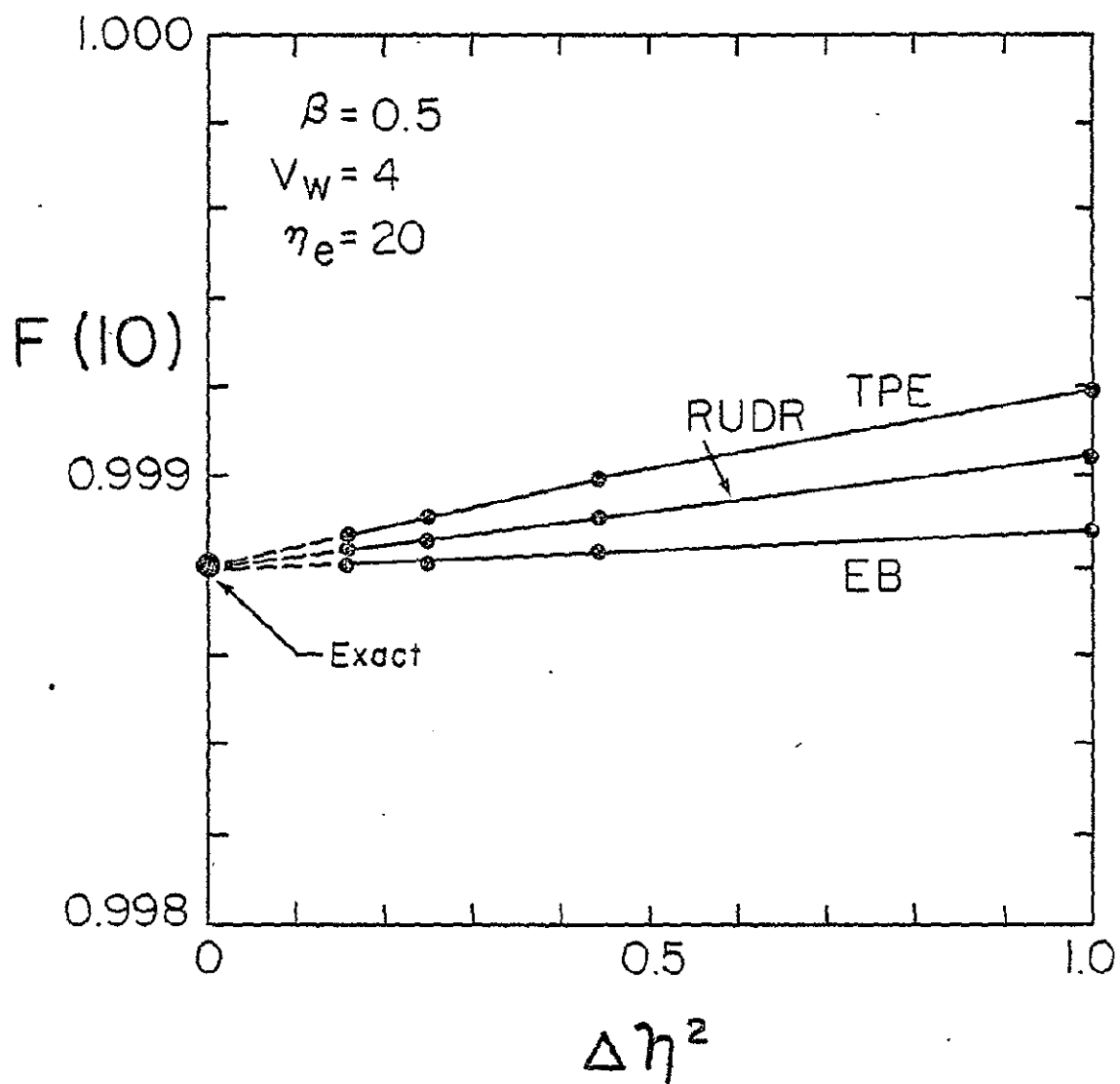


FIGURE 16: MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION
 POINT-STEP SIZE STUDY FOR FINITE DIFFERENCE
 SCHEMES, F AT $\eta = 10.0$



ORIGINAL PAGE IS
 OF POOR QUALITY

FIGURE 17: MODERATE INJECTION AT AN AXISYMMETRIC STAGNATION POINT-
 STEP SIZE STUDY FOR THE EXPONENTIAL SCHEMES, F AT $\eta = 10.0$

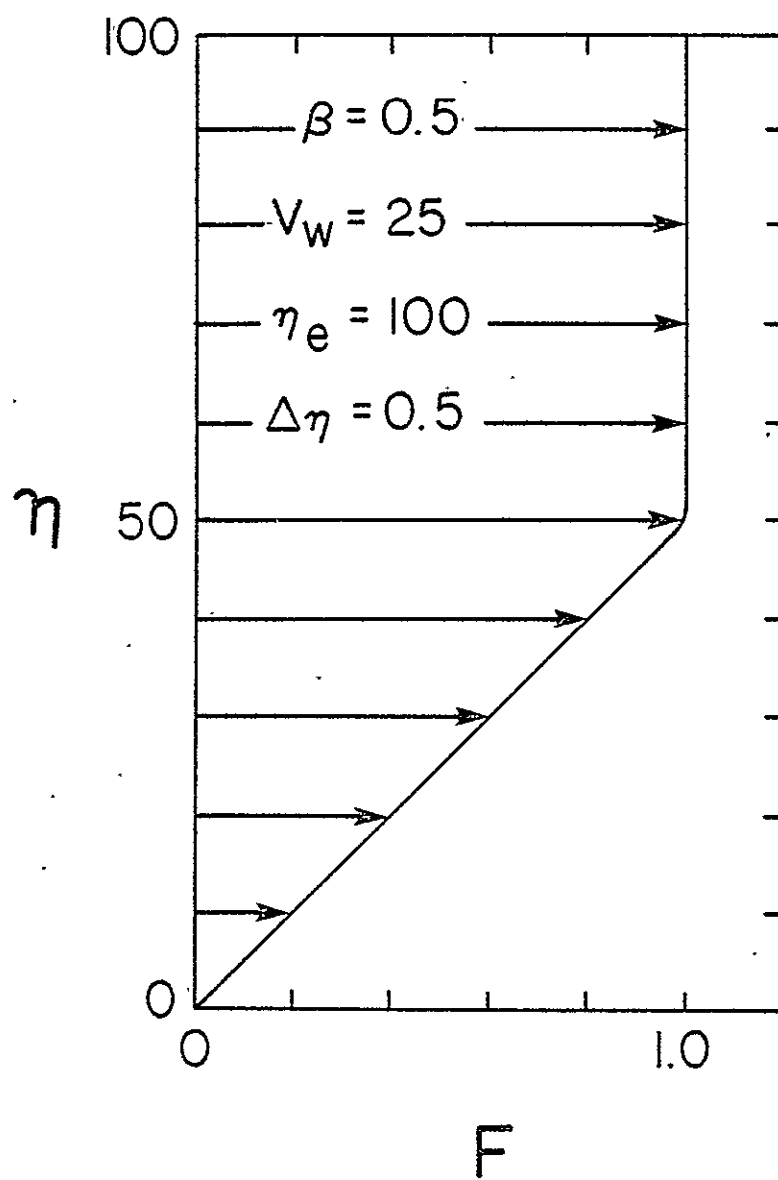


FIGURE 18: STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT-VELOCITY PROFILE

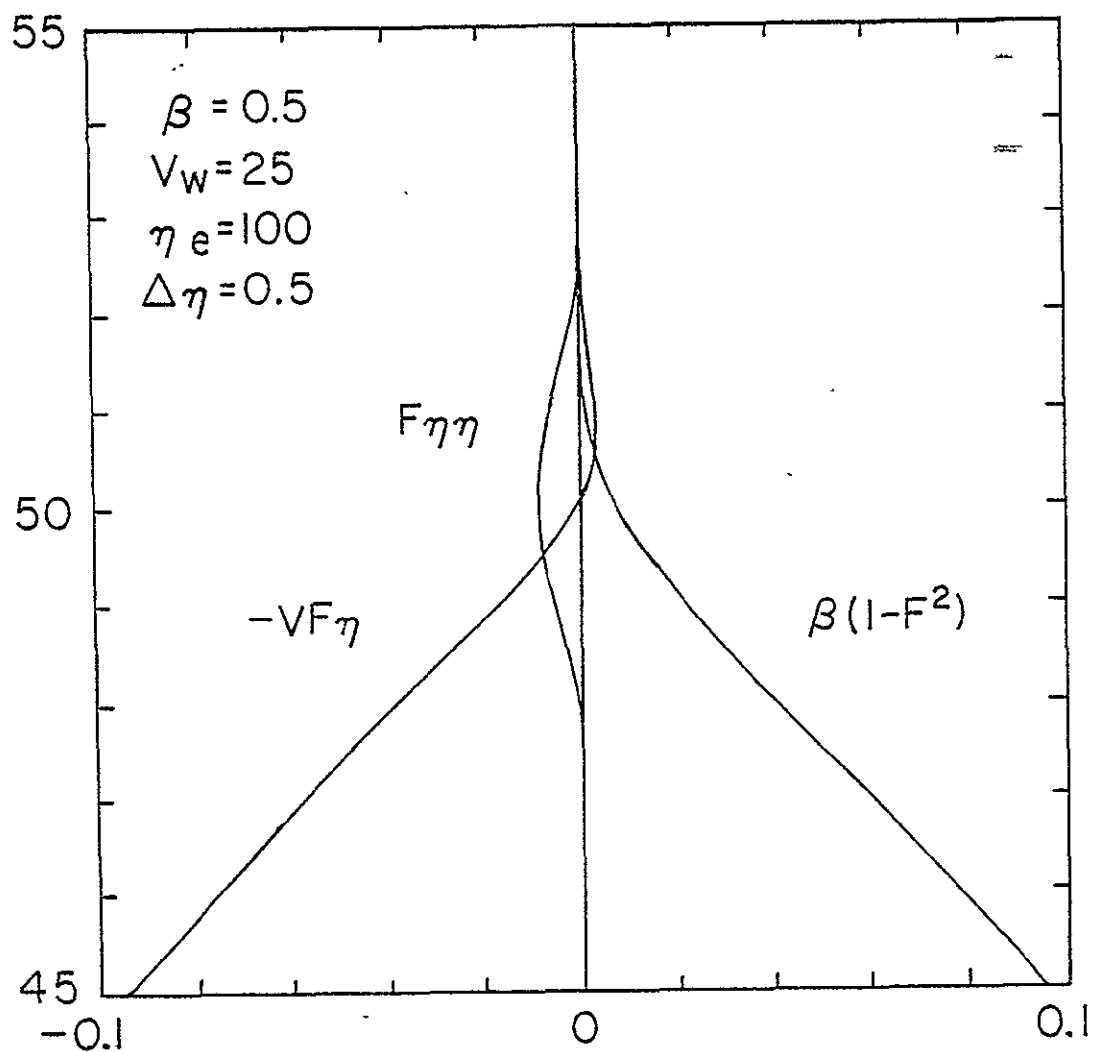


FIGURE 19: STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT-MOMENTUM BALANCE

ORIGINAL PAGE IS
OF POOR QUALITY

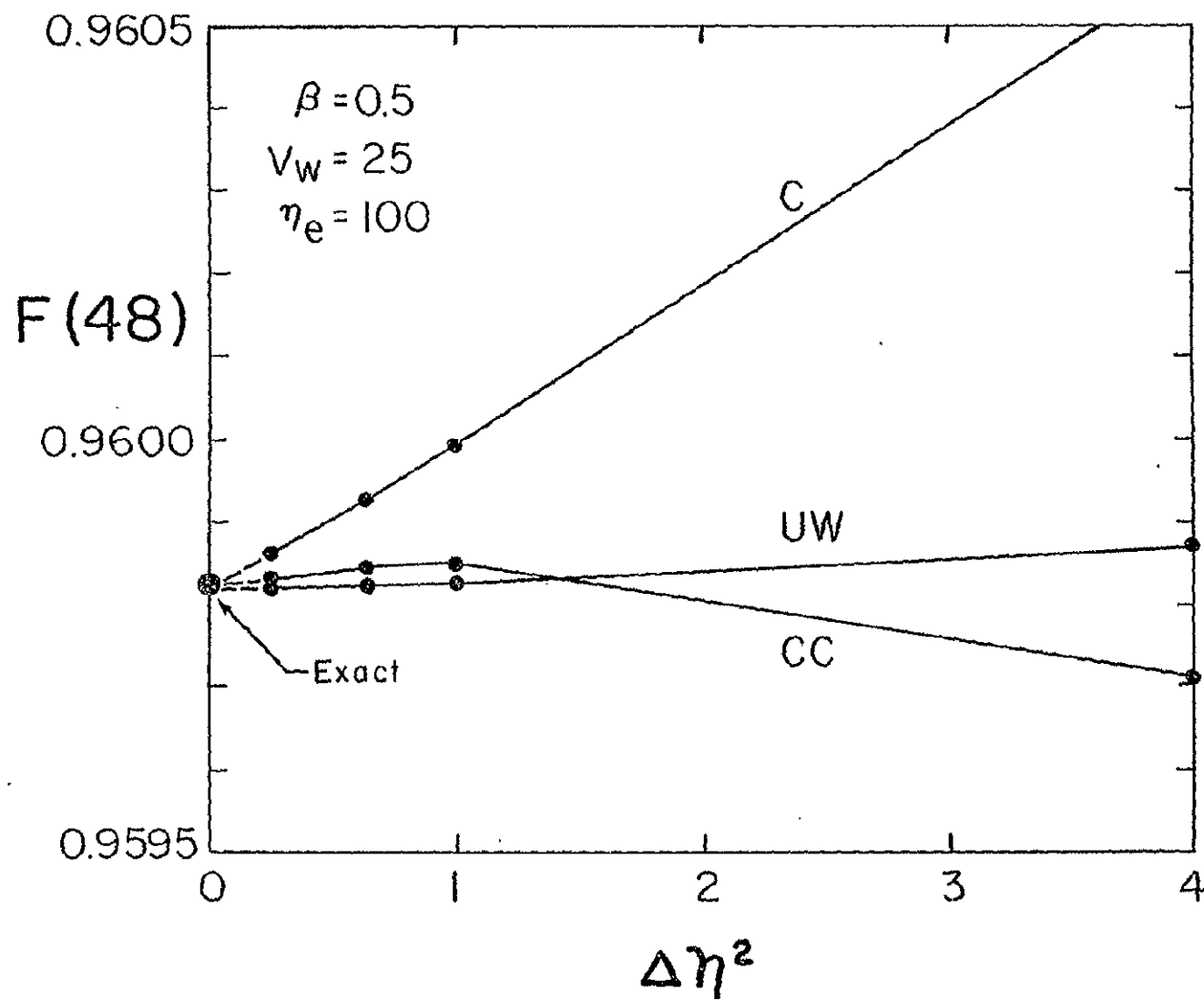


FIGURE 20: STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT-STEP SIZE STUDY FOR FINITE DIFFERENCE SCHEMES, F AT $\eta = 48$

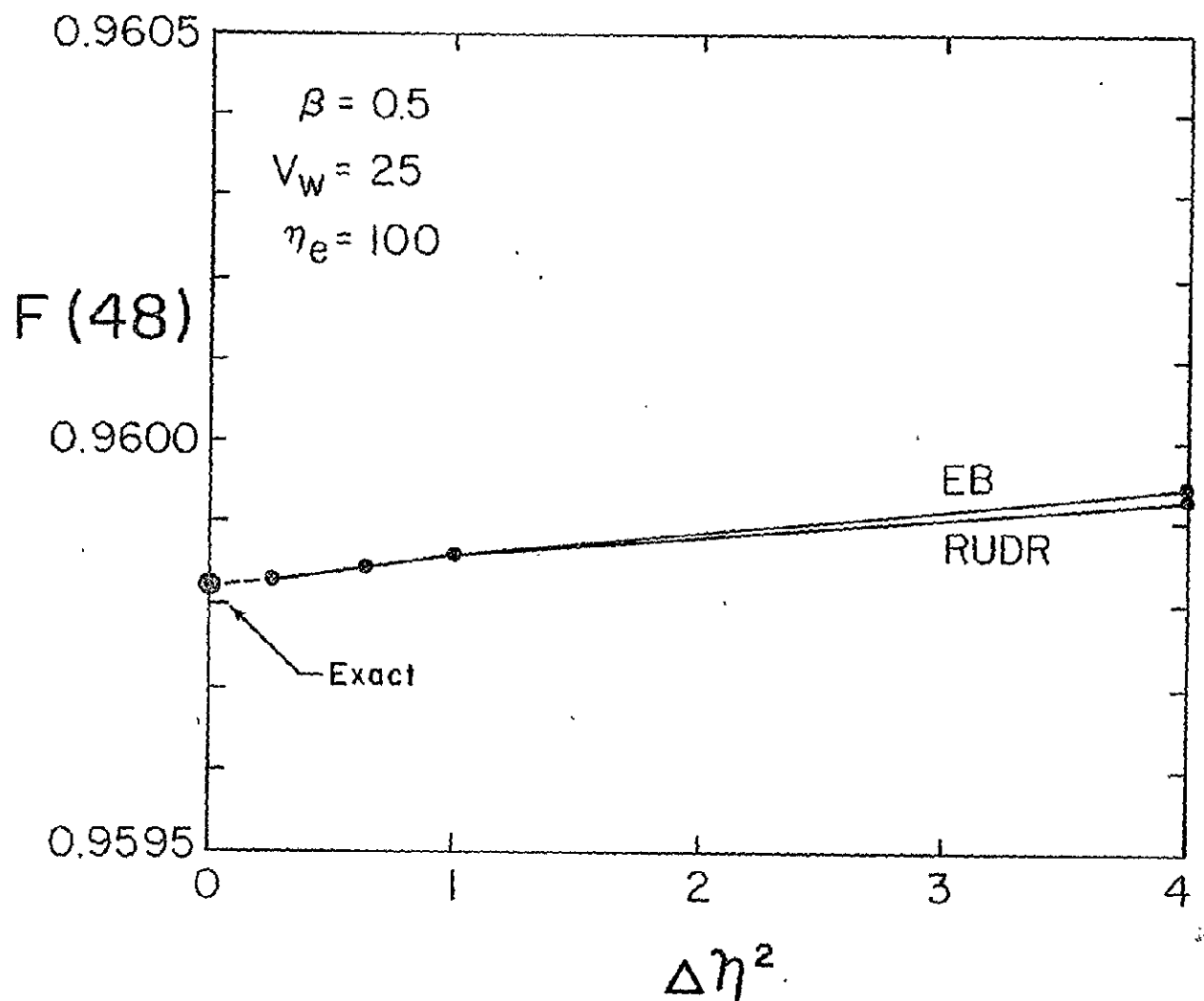


FIGURE 21: STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT-STEP SIZE STUDY FOR THE EXPONENTIAL SCHEMES, F AT $\eta = 48$

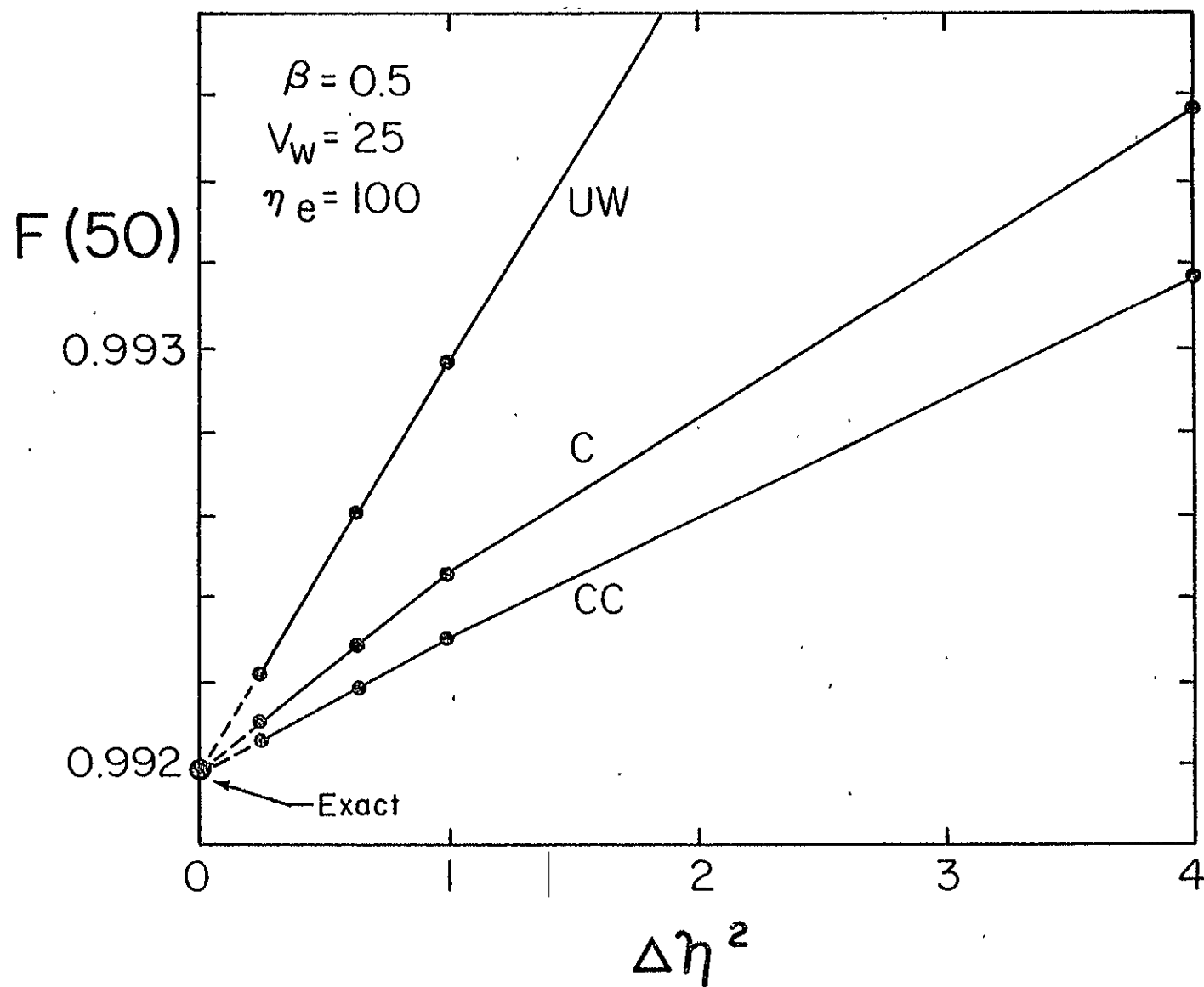


FIGURE 22: STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT-STEP SIZE STUDY FOR FINITE DIFFERENCE SCHEMES, F AT $\eta = 50$.

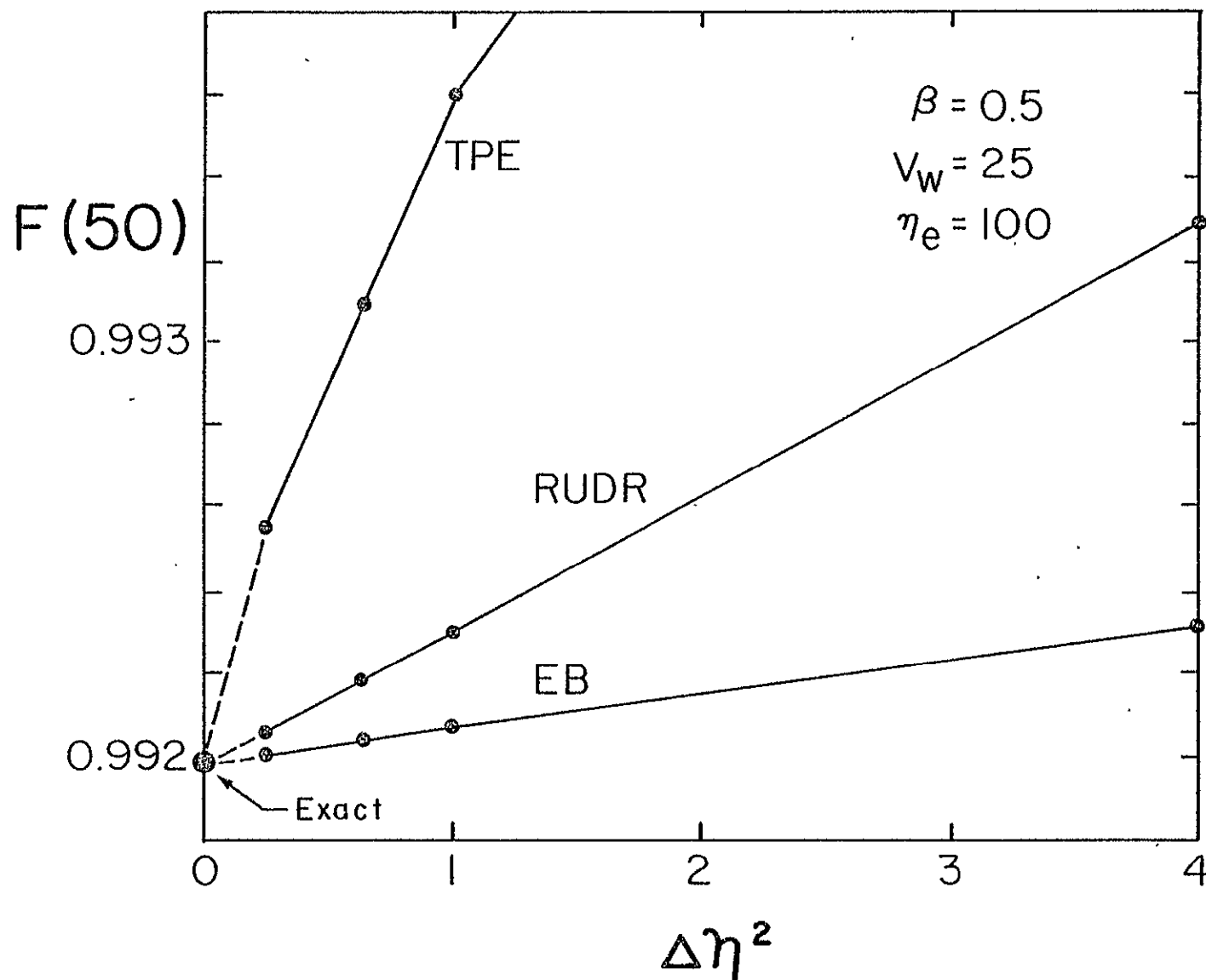


FIGURE 23: STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT-STEP SIZE STUDY FOR THE EXPONENTIAL SCHEMES, F AT $\eta = 50$.

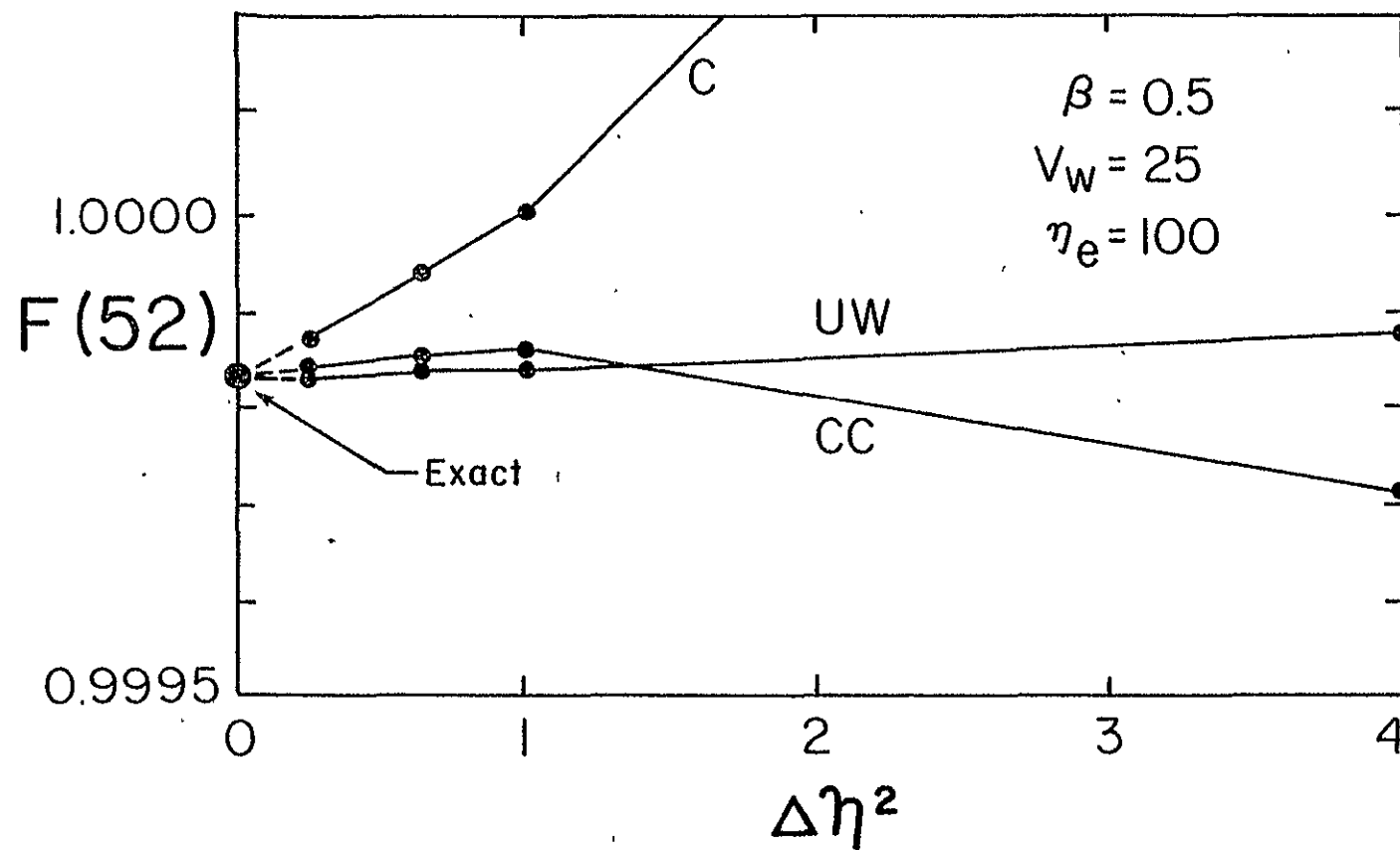


FIGURE 24: STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT-STEP SIZE STUDY FOR FINITE DIFFERENCE SCHEMES, F AT $\eta = 52$

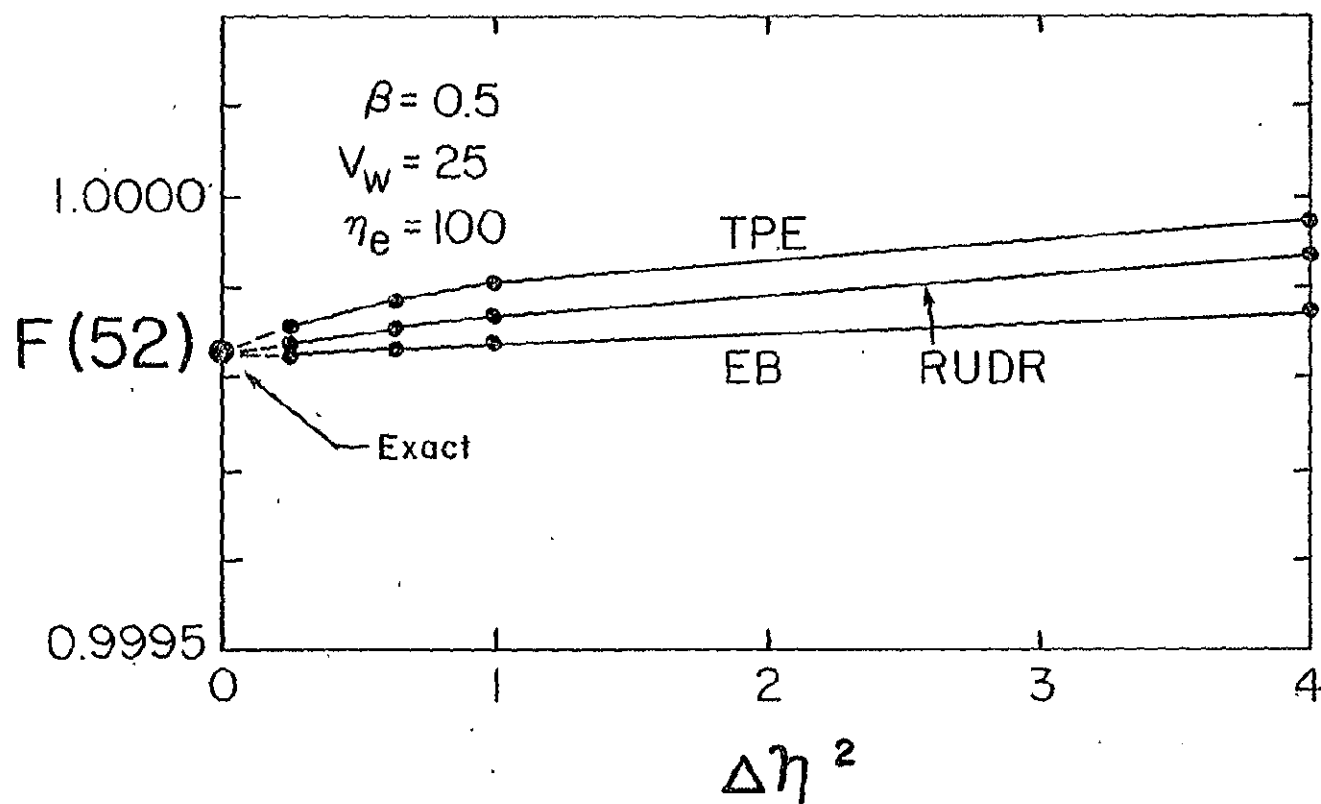


FIGURE 25: STRONG BLOWING AT AN AXISYMMETRIC STAGNATION POINT-STEP SIZE STUDY FOR THE EXPONENTIAL SCHEMES, F AT $\eta = 52$